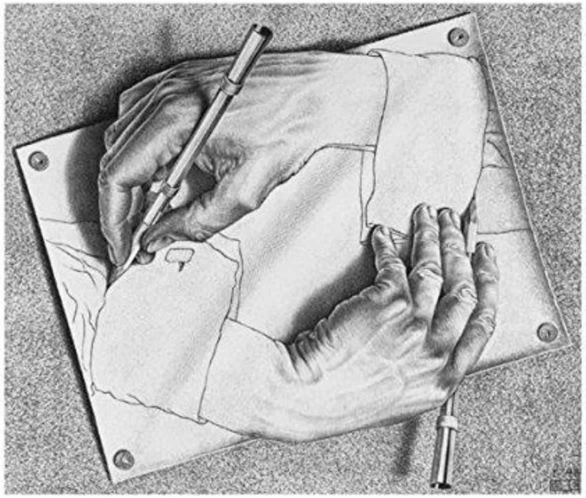
University of Ljubljana, Faculty of Computer and Information Science

Computational Complexity



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Analysis of Algorithms and Heuristic Problem Solving

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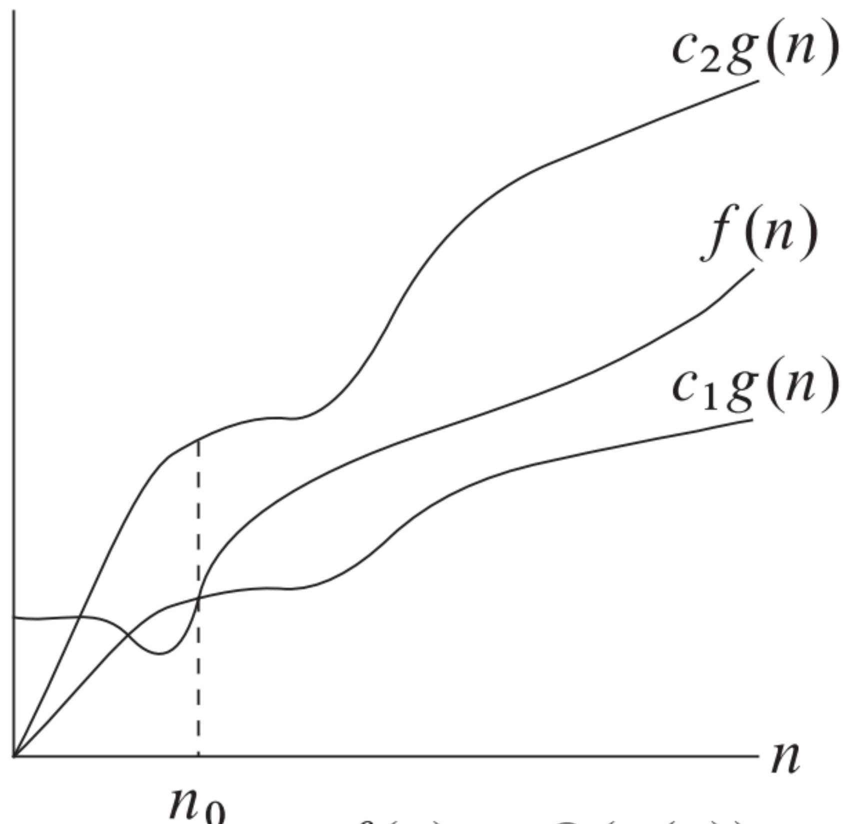
Asymptotically tight bound Θ

- Given function g(n), we denote with $\Theta(g(n))$ a set of functions:

- $\Theta(g(n)) = \{ f(n); \exists c_1, c_2, n_0 > 0, \forall n > n_0: \ 0 \le c_1 g(n) \le f(n) \le c_2 g(n)\}$

- notation used is $f(n) \in \Theta(g(n))$ and more frequently $f(n) = \Theta(g(n))$
- g(n) is asymptotically tight bound for f(n)
- assumption: g(n) is asymptotically positive function

Θ(g(n))



$f(n) = \Theta(g(n))$

An example

- Let us show that ½ $n^2 - 3n = \Theta(n^2)$
- find c_1, c_2, n_0

- Home work:
  - show that $an^2 + bn + c = \Theta(n^2)$
  - show for all polynomials p(n), $p(n) = \sum_{i=0}^{d} a_i n^i, a_d > 0$, that p(n) = $\Theta(n^d)$
  - show $6n^3 \neq \Theta(n^2)$
- we denote constant function as $\Theta(n^0) = \Theta(1)$

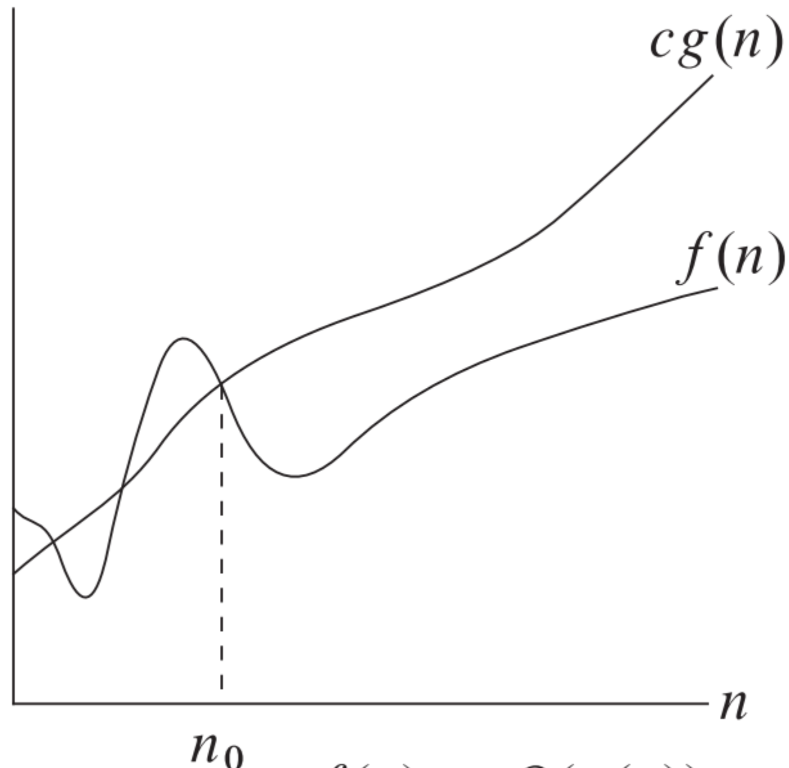
Asymptotical upper bound O

- for functions g(n) we write O(g(n)) to be a set of functions for which the following holds:

- $O(g(n)) = \{ f(n);\ \exists c, n_0 > 0, \forall n > n_0: 0 \le f(n) \le cg(n) \}$

- we use notation $f(n) \in O(g(n))$ or more frequently $f(n) = O(g(n))$
- g(n) is asymptotical upper bound for f(n)
- attention! the literature tend to be imprecise in this notation
- use also as an anonymous function, for example
  $T(n) = 2\ T(n/2) + O(n)$

O(g(n))



$f(n) = O(g(n))$

Alternative definitions

- for upper bound

$$f(n) = O\big(g(n)\big) \Leftrightarrow \lim_{n \to \infty} \frac{|f(n)|}{g(n)} < \infty \text{ and the limit exists}$$

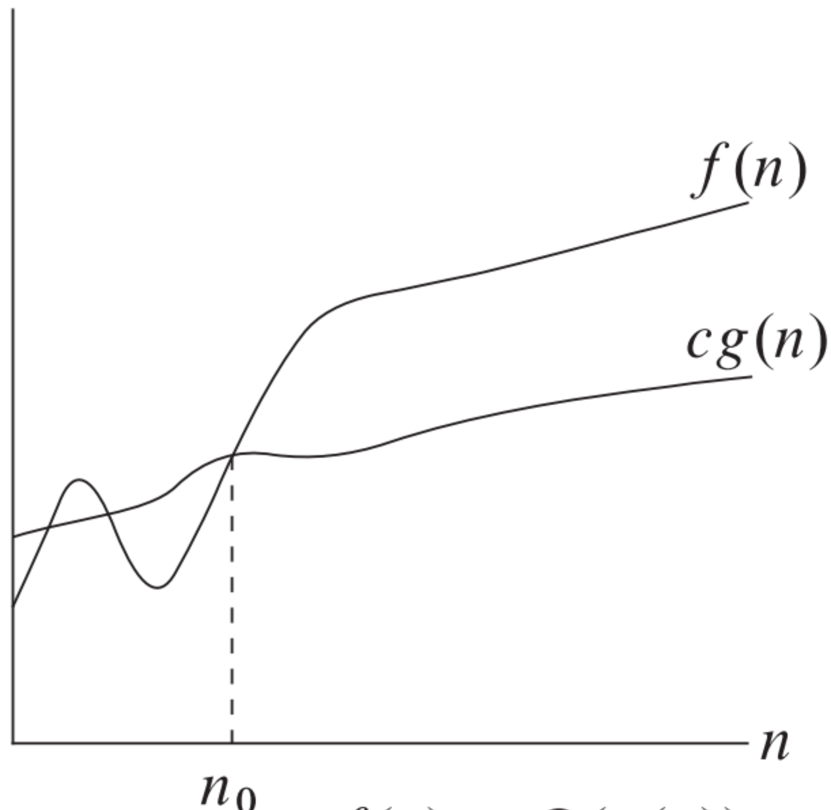
Examples

- Show $\frac{1}{2} n^2 - 3n = O(n^2)$
- Show at home:
  - $an^2 + bn + c = O(n^2)$
  - $an + c = O(n^2)$

Asymptotical lower bound Ω

- For function g(n) we write $\Omega(g(n))$ to be a set of functions:
- $\Omega(g(n)) = \{ f(n); \exists c, n_0 > 0, \forall n > n_0: 0 \leq cg(n) \leq f(n) \}$
- notation $f(n) \in \Omega(g(n))$ or more frequently $f(n) = \Omega(g(n))$
- g(n) is asymptotical lower bound for f(n)
- attention, the literature might be imprecise

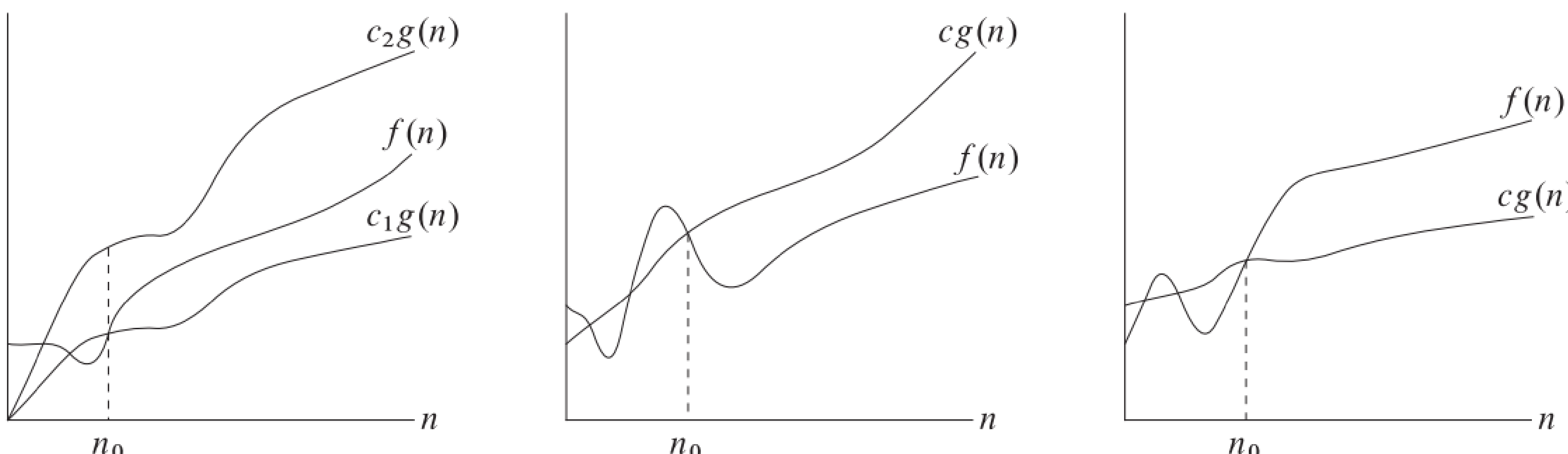
Ω(g(n))



$f(n) = \Omega(g(n))$

Relations between asymptotical bounds

- for functions f(n) and g(n) it holds:
- f(n) = Θ(g(n)) if and only if f(n) = O(g(n)) and f(n) = Ω(g(n))



$f(n) = \Theta(g(n))$ $f(n) = O(g(n))$ $f(n) = \Omega(g(n))$

Imprecise boundaries, notations o and ω

- $o(g(n)) = \{ f(n); \forall c > 0, \exists n_0 > 0, \forall n > n_0: 0 \leq f(n) < cg(n) \}$
- e.g., $7n = o(n^2)$ in $3n^2 \neq o(n^2)$
- $o(g(n))$ is an imprecise upper bound
- $f(n) = o(g(n)) \leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

- $\omega(g(n)) = \{ f(n); \forall c > 0, \exists n_0 > 0, \forall n > n_0: 0 \leq cg(n) < f(n) \}$
- e.g., $n^2 = \omega(n)$ and $3n \neq \omega(n)$
- $\omega(g(n))$ is an imprecise lower bound
- $f(n) = \omega(g(n)) \leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

Properties of asymptotic bounds1/2

- transitivity

$f(n) = \Theta(g(n)) \wedge g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$

$f(n) = O(g(n)) \wedge g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$

$f(n) = \Omega(g(n)) \wedge g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$

$f(n) = o(g(n)) \wedge g(n) = o(h(n)) \Rightarrow f(n) = o(h(n))$

$f(n) = \omega(g(n)) \wedge g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$

- reflexivity

$f(n) = \Theta(f(n))$

$f(n) = O(f(n))$

$f(n) = \Omega(f(n))$

Properties of asymptotic bounds 2/2

- symmetry

$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

- transpose symmetry

$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$

$f(n) = o(g(n)) \Leftrightarrow g(n) = \omega(f(n))$

- analogy with numbers

$f(n) = O(g(n))$ $a \leq b$

$f(n) = \Omega(g(n))$ $a \geq b$

...

- but not trichotomy

e.g., between two numbers exactly one of the following relations holds $a<b$, $a=b$, $a>b$

why not for asymptotic function bounds?

Divide and conquer algorithms

- Idea:
  - **divide** the problem into several (equal) parts
  - (recursively) **conquer (solve)** each of the sub problems
  - **combine** sub problem solutions

- An example: maximum subarray problem

Maximum subarray problem

```
// maximal subarray of array A[low…high] crossing the point mid
findMaxCrossingSubarray(A, low, mid, high) {
    leftSum = -∞ ; sum = 0 ;
    for (i = mid ; i >= low ; i--) {
        sum = sum + A[i] ;
        if (sum > leftSum) {
            leftSum = sum ;
            maxLeft = i ;
        }
    }
    rightSum = -∞ ; sum = 0 ;
    for (j = mid +1; j <= high ; j++) {
        sum = sum + A[j] ;
        if (sum > rightSum) {
            rightSum = sum ;
            maxRight = j ;
        }
    }
    return (maxLeft, maxRight, leftSum + rightSum) ;
}
```

```
// maximal subarray of array A[low…high]

findMaxSubarray(A, low, high) {
    if (low == high) // boundary condition
        return (low, high, A[low]) ;

    else {
        mid = (low + high) / 2 ;
        (leftLow, leftHigh, leftSum) = findMaxSubarray(A, low, mid) ;
        (rightLow, rightHigh, rightSum) = findMaxSubarray(A, mid+1, high) ;
        (crossLow, crossHigh, crossSum) = findMaxCrossingSubarray(A, low, mid, high) ;
        if (leftSum >= rightSum && leftSum >= crossSum)
            return (leftLow, leftHigh, leftSum) ;
        else if (rightSum >= leftSum && rightSum >= crossSum)
            return (rightLow, rightHigh, rightSum) ;
        else return (crossLow, crossHigh, crossSum) ;
    }
}
```

Kadane algorithm

- idea: for each position compute the maximum subarray result for the subarray ending at given position

```
findMaxSubarrayKadane(A) {
    maxEndingHere = 0 ;
    maxSoFar = 0 ;
    for (i=1 ; i <= A.length ; i ++) {
        maxEndingHere = max(0, maxEndingHere + A[i]) ;
        maxSoFar = max(maxSoFar, maxEndingHere) ;
    }
    return maxSoFar ;
}
```

Four approaches to the analysis of divide-and-conquer algorithms

- substitution method:
  - guess the solution
  - using induction find the constants and prove the solution is valid (requires some practice and knowledge of some tricks)
- recursive tree:
  - draw recursion tree and sum complexity level-wise and altogether;
  - prove with induction that the result is correct
- master theorem
- Akra-Bazzi theorem

Master theorem

- for divide and conquer algorithms
- assume constants $a \geq 1, b > 1$, a function $f(n)$
- $T(n)$ is defined for nonnegative integers with recurrent equation

$$\mathrm{T(n) = aT(n/b) + f(n)},$$

where n/b is either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. T(n) has the following asymptotic bounds

$$\left\{ \begin{array}{ll} T(n) = \theta(n^{\log_b a}) & ; f(n) = O\left(n^{\log_b a - \varepsilon}\right) \text{ for constant } \varepsilon > 0 \\ T(n) = \theta(n^{\log_b a} \log n) & ; f(n) = \theta\left(n^{\log_b a}\right) \\ T(n) = \theta(f(n)) & ; f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \text{ for constant } \varepsilon > 0, \\ & \text{if } af\left(\frac{n}{b}\right) \leq cf(n) \text{, for constant } c < 1, \text{and all large enough } n \end{array} \right\}$$

Using the master theorem

- examples when it works
- and when it doesn‘t

Akra-Bazzi theorem

(Mohamad Akra and Louay Bazzi, 1998)

Let

$$\mathrm{T(x)} = \begin{Bmatrix} \theta(1) & ; 1 \le x \le x_0 \\ \sum_{i=1}^{k} a_i T(b_i x) + f(x) & ; x > x_0 \end{Bmatrix}, \text{ where}$$

- real number $x >= 1$,
- constant $x_0 >= 1/b_i$ and $x_0 >= 1/(1-b_i)$ for $i = 1, 2, \ldots, k$
- a_i is a positive constant for $i = 1, 2, \ldots, k$
- b_i is constant $0 < b_i < 1$ for $i = 1, 2, \ldots, k$
- $k >= 1$ is an integer constant
- $f(x)$ is nonnegative function satisfying polynomial growth condition: there exist positive constants c_1 and c_2 such that for all $x>=1$ *and for* $i = 1, 2, \ldots, k$, for all u *for* which $b_i x <= u <= x$ it holds $c_1 f(x) <= f(u) <= c_2 f(x)$. Alternatively: if $|f'(x)|$ is upper bounded by polynomial of x, then $f(x)$ satisfies polynomial growth condition.
- real number p is the only solution of equation $\sum_{i=1}^{k} a_i b_i^p = 1$

Then the solution of the recursion is

$$\mathrm{T(x)} = \theta\left(x^p\left(1 + \int_1^x \frac{f(u)}{u^{p+1}} du\right)\right).$$

Akra-Bazzi theorem – the strong form

Let

$$\mathrm{T(x)} = \begin{cases} \theta(1) & ; 1 \le x \le x_0 \\ \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) + f(x) & ; x > x_0 \end{cases}, \text{ where}$$

- real number $x >= 1$,
- constant $x_0 >= \max(b_i, 1/b_i)$ for $i = 1, 2, \ldots, k$
- a_i is a positive constant for $i = 1, 2, \ldots, k$
- b_i is constant $0 < b_i < 1$ for $i = 1, 2, \ldots, k$
- $k >= 1$ is an integer constant
- $|f(x)| = O(x^c)$ for any $c \in N$
- $|h_i(x)| = O(\frac{x}{\log^2 x})$
- real number p is the only solution of equation $\sum_{i=1}^{k} a_i b_i^p = 1$

Then the solution of the recursion is

$$\mathrm{T(x)} = \theta\left(x^p\left(1 + \int_1^x \frac{f(u)}{u^{p+1}} du\right)\right).$$