Linear least squares method. Let $A \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix with $m \geq n$. Let $\mathbf{b} \in \mathbb{R}^m$ be a vector. How would you find the orthogonal projection of $\mathbf{b}$ on to the column space of A, $C(A)$? (Assume that the columns of A are linearly independent.)

1. We want to approximate a real function f on the interval $[a,b]$ with a polynomial. We will do this (perhaps naïvely) by dividing the interval $[a,b]$ with $k+1$ equidistant points $a = x_0, x_1, \ldots, x_k = b$ and then find the coefficients of the polynomial $p(x)$ that is the best fit to the data in the table below according to the linear least squares method.

x_0	x_1	$\cdots$	x_i	$\cdots$	x_k
$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_i)$	$\cdots$	$f(x_k)$

   (a) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n. Write the matrix A of the corresponding linear system and the right-hand side of $\mathbf{b}$ according to the data in the table above.

   (b) Find the approximations of orders 0, 1 and 2 for the function $f(x) = \frac{x^2}{1+x^2}$ on the interval $[-1,1]$ using the points $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

   (c) Using `octave` approximate the function $g(x) = \frac{1}{1+25x^2}$ on $[-1,1]$ with polynomials of order $0, 2, \ldots, 20$, dividing the interval $[-1,1]$ with 21 equidistant points. Find the approximations for the exact data and for data with (artificially added) errors. Using the `plot` command plot the graphs of the original functions and all the approximations.

2. Use the linear least squares method to solve the following problem: In the plane $\mathbb{R}^2$ we have n transmitters at known locations $(p_1,q_1),\ldots,(p_n,q_n)$. A receiver can measure the distances $d_1,\ldots,d_n$ from these transmitters. Given those distances, we would like to determine the position of the receiver. In the ideal case the measurements are exact and for each $i = 1,\ldots,n$ we have an equation

$$(x-p_i)^2 + (y-q_i)^2 = d_i^2.$$

   The solution of this system of equations then determines the unknown position of the receiver (x,y).

   (a) The first problem is that the equations are *not* linear. But the difference of two consecutive equations is a linear equation. Write down these differences to obtain a system of $n-1$ linear equations.

   (b) Write the matrix $A \in \mathbb{R}^{(n-1)\times 2}$ of the system and the corresponding right-hand side $\mathbf{b} \in \mathbb{R}^{n-1}$. Additional problem is that the measurements are not exact which means that the system $A\mathbf{x} = \mathbf{b}$ (almost surely) has no solution.

   (c) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$. Write an `octave` function `X = sprejemnik([pi, qi], [di])` that finds the position of the receiver $X(x,y)$ given transmitter positions (p_i,q_i) and distances d_i. (The positions (p_i,q_i) are contained in an $n \times 2$ matrix and the distances d_i are given in a column matrix of length n. The result X should be row vector `X = [x, y]`.)

(d) Test the function using artificial data from the files `oddajniki.txt` and `razdalje.txt` found on *ucilnica*. You can import them into octave using the `load` command. Visualize the results.