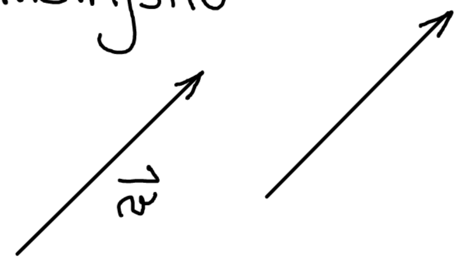
Vaje MAT VSP, 27.12.2022

VEKTORJI

geometrijsko:

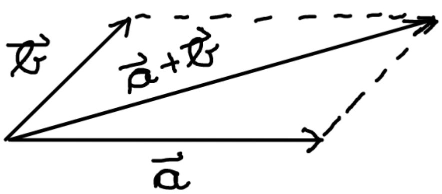


računsko:

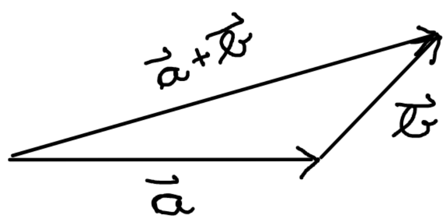
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Osnovne operacije:

• seštevanje



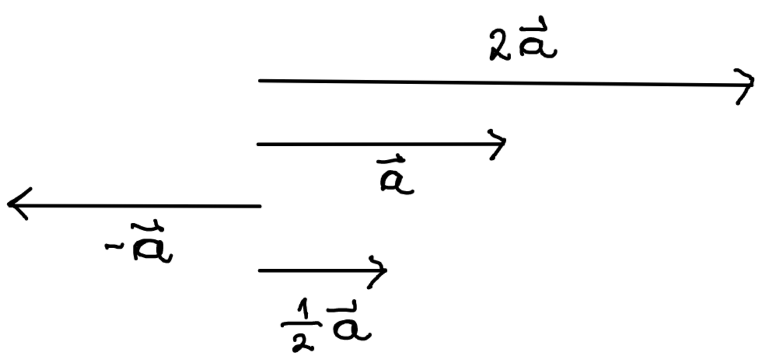
paralelogramsko pravilo



trikotniško pravilo

$$\vec{a}+\vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_m+b_m \end{bmatrix}$$

• množenje vektorja s skalarjem:



$$\alpha \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_m \end{bmatrix}$$

(vzporedna)
Vektorja $\vec{a}$ in $\vec{b}$ sta kolinearna, če obstaja skalar α, da velja:

$$\vec{a} = \alpha \vec{b} \quad \text{ali} \quad \vec{b} = \alpha \vec{a}.$$

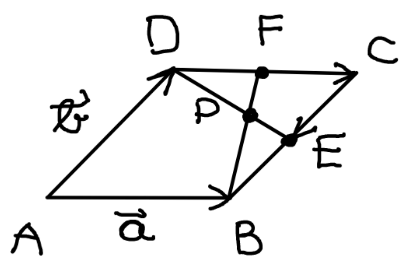
Če sta $\vec{a}, \vec{b} \in \mathbb{R}^2$ nekolinearna vektorja, potem lahko vsak $\vec{c} \in \mathbb{R}^2$ zapišemo kot

$$\vec{c} = \alpha \cdot \vec{a} + \beta \cdot \vec{b}.$$

1. V rombu z oglišči $ABCD$ označimo z E točko na razpolovišču stranice BC in z F točko na razpolovišču stranice CD. Naj bo P točka, v kateri se sekata daljici BF in DE.

   (a) Kolikšno je razmerje med dolžinama daljic DP in PE?

   (b) Prepričaj se, da točka P leži na diagonali AC. V kolikšnem razmerju ta točka deli diagonalo?



a) $|\overrightarrow{DP}| : |\overrightarrow{PE}| = ?$

$$\overrightarrow{DP} = k \cdot \overrightarrow{DE} \qquad \overrightarrow{DP} = -\vec{b} + \vec{a} + k \cdot \overrightarrow{BF}$$

$$\overrightarrow{DP} = k \cdot (\vec{a} - \tfrac{1}{2}\vec{b}) \qquad \overrightarrow{DP} = -\vec{b} + \vec{a} + k(\vec{b} - \tfrac{1}{2}\vec{a})$$

$$\overrightarrow{DP} = \overrightarrow{DP}$$

$$k(\vec{a} - \tfrac{1}{2}\vec{b}) = -\vec{b} + \vec{a} + k(\vec{b} - \tfrac{1}{2}\vec{a})$$

$$k(\vec{a} - \tfrac{1}{2}\vec{b}) - k(\vec{b} - \tfrac{1}{2}\vec{a}) = \vec{a} - \vec{b}$$

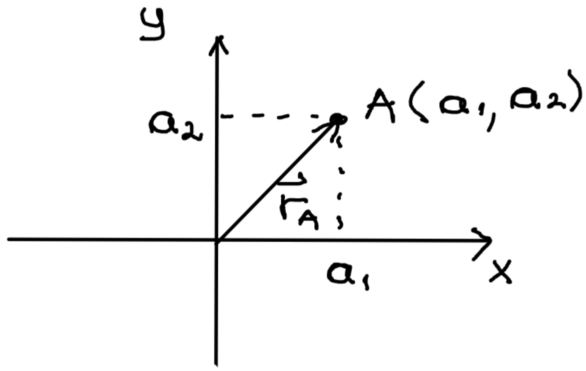
$$k(\vec{a} - \tfrac{1}{2}\vec{b} - \vec{b} + \tfrac{1}{2}\vec{a}) = \vec{a} - \vec{b}$$

$$k(\tfrac{3}{2}\vec{a} - \tfrac{3}{2}\vec{b}) = \vec{a} - \vec{b}$$

$$k = \frac{\cancel{\vec{a} - \vec{b}}}{\frac{3}{2}(\cancel{\vec{a} - \vec{b}})} = \frac{1}{\frac{3}{2}} = \underline{\underline{\frac{2}{3}}}$$

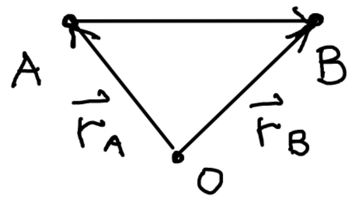
$\overrightarrow{DP} = \frac{2}{3}\overrightarrow{DE}$

$|\overrightarrow{DP}| : |\overrightarrow{PE}| = 2 : 1$



krajevni vektor točke A:

$$\vec{r}_A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

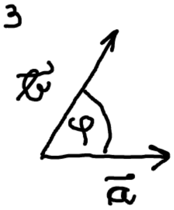


$$\overrightarrow{AB} = -\vec{r}_A + \vec{r}_B = \vec{r}_B - \vec{r}_A$$

Skalarni produkt vektorjev $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$:

$$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$$



$$\Downarrow$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

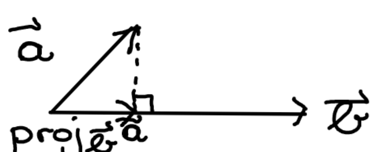
φ je kot med vektorjema $\vec{a}$ in $\vec{b}$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

dolžina vektorja $\vec{a}$

$$\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$$

Pravokotna projekcija vektorja $\vec{a}$ na vektor $\vec{b}$:



$\text{proj}_{\vec{b}}\vec{a}$

$$\mathrm{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b}$$

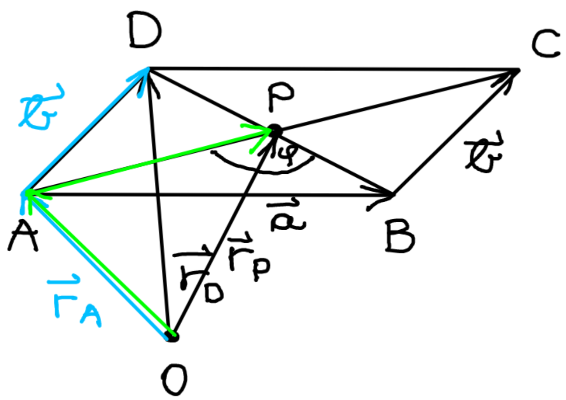
3. Dan je paralelogram $ABCD$ z oglišči $A(-3,-2,0)$, $B(3,-3,1)$, $C(5,0,2)$.

   (a) Določi oglišče D in presečišče diagonal.

   (b) Izračunaj dolžini stranic paralelograma $ABCD$ in kot med njegovima diagonalama.

   (c) Izračunaj ploščino paralelograma $ABCD$.

a)



$$\vec{a} = \overrightarrow{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix}3\\-3\\1\end{bmatrix} - \begin{bmatrix}-3\\-2\\0\end{bmatrix} = \begin{bmatrix}6\\-1\\1\end{bmatrix}$$

$$\vec{b} = \overrightarrow{BC} = \vec{r}_C - \vec{r}_B = \begin{bmatrix}5\\0\\2\end{bmatrix} - \begin{bmatrix}3\\-3\\1\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$\vec{r}_D = \vec{r}_A + \vec{b} = \begin{bmatrix}-3\\-2\\0\end{bmatrix} + \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\boxed{D(-1,1,1)}$$

diagonali v paralelogramu se razpolavljata

$$\vec{r}_P = \vec{r}_A + \frac{1}{2}\overrightarrow{AC} = \vec{r}_A + \frac{1}{2}(\vec{a}+\vec{b}) =$$

$$= \begin{bmatrix}-3\\-2\\0\end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix}6\\-1\\1\end{bmatrix} + \begin{bmatrix}2\\3\\1\end{bmatrix}\right) =$$

$$= \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{P(1,-1,1)}$$

b) Dolžini stranic:

$$|\vec{AB}| = |\vec{a}| = \sqrt{6^2+(-1)^2+1^2} = \underline{\underline{\sqrt{38}}}$$

$$|\vec{BC}| = |\vec{b}| = \sqrt{2^2+3^2+1^2} = \underline{\underline{\sqrt{14}}}$$

Kot med diagonalama:

$$\varphi = \angle(\vec{PA}, \vec{PB}) = ?$$

$$\vec{PA} = \vec{r}_A - \vec{r}_P = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{PB} = \vec{r}_B - \vec{r}_P = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\cos\varphi = \frac{\vec{PA}\cdot\vec{PB}}{|\vec{PA}|\cdot|\vec{PB}|} = \frac{(-4)\cdot 2+(-1)(-2)+(-1)\cdot 0}{\sqrt{(-4)^2+(-1)^2+(-1)^2}\cdot\sqrt{2^2+(-2)^2+0^2}}$$

$$= \frac{-6}{\sqrt{18}\cdot\sqrt{8}} = \frac{-\cancel{6}}{\cancel{3}\sqrt{2}\cdot\cancel{2}\sqrt{2}} = -\frac{1}{2}$$

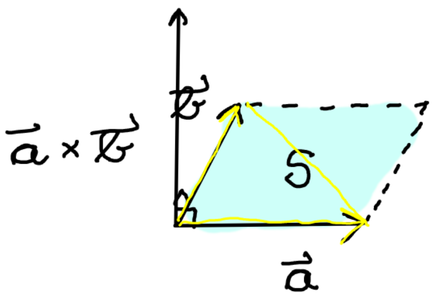
$$\Rightarrow \varphi = \arccos\left(-\frac{1}{2}\right) = 120^\circ$$

Vektorski produkt vektorjev $\vec{a}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ in $\vec{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$:

$$\vec{a}\times\vec{b}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}a_2b_3-b_2a_3\\a_3b_1-b_3a_1\\a_1b_2-b_1a_2\end{bmatrix}$$

$a_1 \quad b_1$

$a_2 \quad b_2$



$S=|\vec{a}\times\vec{b}|$

↑ dolžina $\vec{a}\times\vec{b}$

$S_\triangle=\dfrac{|\vec{a}\times\vec{b}|}{2}$

c) $S_\square=|\vec{a}\times\vec{b}|$

$$\vec{a}\times\vec{b}=\begin{bmatrix}6\\-1\\1\end{bmatrix}\times\begin{bmatrix}2\\3\\1\end{bmatrix}=\begin{bmatrix}(-1)\cdot1-3\cdot1\\1\cdot2-1\cdot6\\6\cdot3-2\cdot(-1)\end{bmatrix}=\begin{bmatrix}-4\\-4\\20\end{bmatrix}$$

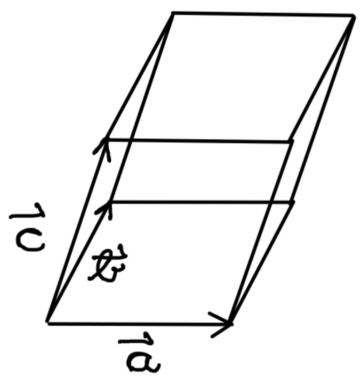
$6 \quad 2$

$-1 \quad 3$

$$S=|\vec{a}\times\vec{b}|=\sqrt{(-4)^2+(-4)^2+20^2}=\underline{\underline{\sqrt{432}}}$$

Mešani produkt vektorjev $\vec{a}$, $\vec{b}$ in $\vec{c}$:

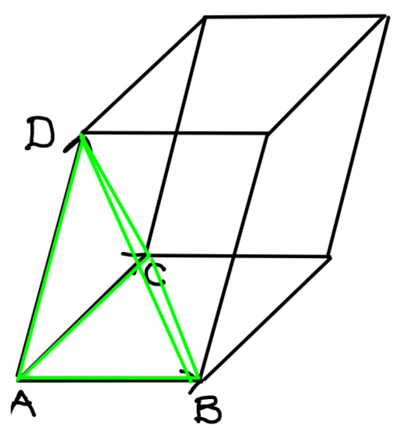
$$(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$



Absolutna vrednost
$|(\vec{a}, \vec{b}, \vec{c})| =$
$V_{paralelepipeda}$

6. Dane so točke $A(1,1,2)$, $B(1,4,-1)$, $C(3,3,2)$ in $D(4,-1,4)$.
   (a) Izračunaj prostornino paralelepipeda, ki je napet na vektorje AB, AC in AD.
   (b) Izračunaj prostornino piramide $ABCD$.

a)



$$\vec{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

$$\vec{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{AD} = \vec{r}_D - \vec{r}_A = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

$$(\vec{AB}, \vec{AC}, \vec{AD}) = \vec{AB} \cdot (\vec{AC} \times \vec{AD}) =$$

$$= \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ -10 \end{bmatrix} = 0 \cdot 4 + 3 \cdot (-4) + (-3) \cdot (-10) =$$

$$= 0 - 12 + 30 = 18$$

$$V_{paralelepipeda} = |(\vec{AB}, \vec{AC}, \vec{AD})| = |18| = \underline{\underline{18}}$$

$$\vec{AC} \times \vec{AD} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -10 \end{bmatrix}$$

$$\begin{matrix} 2 & 3 \\ 2 & -2 \end{matrix}$$

b) $$V_{piramide} = \frac{1}{6} V_{paralelepipeda} = \frac{1}{6} \cdot 18 = \underline{\underline{3}}$$

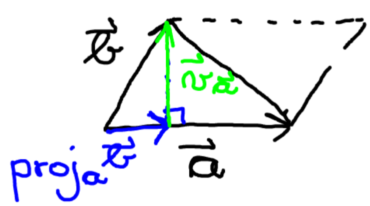
4. (a) Izračunaj kot med vektorjema $\vec{a} = [2, -2, 4]^T$ in $\vec{b} = [2, 4, -2]^T$.

   (b) Kolikšna je ploščina trikotnika, ki ga ta dva vektorja določata?

   (c) Poišči pravokotno projekcijo vektorja $\vec{b}$ na vektor $\vec{a}$ in še vektor, ki v danem trikotniku predstavlja višino na $\vec{a}$.

a) $$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

b)



$$S_\Delta = \frac{1}{2} \cdot |\vec{a} \times \vec{b}|$$

c) $$proj_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a} = \frac{2 \cdot 2 + (-2) \cdot 4 + 4 \cdot (-2)}{\left(\sqrt{4+4+16}\right)^2} \cdot \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

$$= \frac{-12}{24} \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{v}_{\vec{a}} = -proj_{\vec{a}} \vec{b} + \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$