Mathematical modelling, Exam 2

5. 7. 2019

1. The system of equations $2x - y + z = 3$ and $-x + 2y - z = 1$ can be expressed in the form $Ax = b$.

   (a) Find the Moore-Penrose inverse of A, $A^\dagger$.

   (b) Describe the property uniquely characterizing the point $A^\dagger b$ with respect to the system.

   (c) Construct any single matrix, which has the following matrices as their generalized inverses: $\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

Solution.

(a) The matricial form of the system is the following:

$$\underbrace{\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{b}.$$

Since $\operatorname{rank} A = 2$, also $\operatorname{rank}(AA^T) = 2$ and hence $A^\dagger$ is equal to

$$\begin{aligned} A^\dagger = A^T(AA^T)^{-1} &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{6}{11} & \frac{5}{11} \\ \frac{5}{11} & \frac{6}{11} \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{11} & \frac{4}{11} \\ \frac{4}{11} & \frac{7}{11} \\ \frac{1}{11} & -\frac{1}{11} \end{bmatrix}. \end{aligned}$$

(b) Since $A \in \mathbb{R}^{2\times 3}$ and $\operatorname{rank} A = 2$, it follows that the system $Ax = b$ is solvable and the kernel of A is one-dimensional. Hence, there is a one-dimensional family of solutions of the system $Ax = b$. The vector $A^\dagger b$ is the solution of the system of the smallest norm among all solutions.

(c) The matrix A is of size 4×2. By construction of some generalized inverses, the matrix

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

is a generalized inverse of any matrix of the form

$$\begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ X \end{bmatrix},$$

where $X \in \mathbb{R}^{2\times 2}$ is any matrix, and the matrix

$$\begin{bmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

is a generalized inverse of any matrix of the form

$$\begin{bmatrix} Y \\ \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \end{bmatrix},$$

where $Y \in \mathbb{R}^{2\times 2}$ is any matrix. Hence,

$$A = \begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}.$$

2. Given the parametric curve $\gamma(t) = [2\cos(t), 2\sin(t), -t]^\mathsf{T}$:

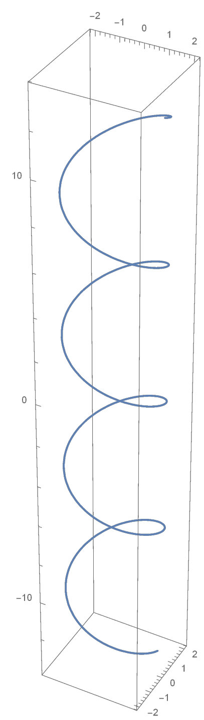
   (a) Sketch/describe γ.

   (b) Parameterize γ with a natural parameter.

   (c) Find the length of γ between points $(2, 0, 0)$ and $(2, 0, 2\pi)$.

Solution.

(a) The sketch of γ is the following:



(b) The natural parameter $s(t)$, which measures the arc length between the points $\gamma(0)$ and $\gamma(t)$ is

$$\begin{aligned} s(t) = \int_0^t \|\gamma'(u)\| \; du &= \int_0^t \|(-2\sin u, 2\cos u, -1\| \; du \\ &= \int_0^t \sqrt{4\sin^2 u + 4\cos^2 u + 1}\, du \\ &= \int_0^t \sqrt{5}\, du = t\sqrt{5}. \end{aligned}$$

Hence, $t(s) = \frac{s}{\sqrt{5}}$ and the parametrization of the curve with the natural parameter s is

$$\gamma(s) = \left(2\cos\left(\frac{s}{\sqrt{5}}\right), 2\sin\left(\frac{s}{\sqrt{5}}\right), -\frac{s}{\sqrt{5}}\right).$$

(c) The point $(2, 0, 0)$ corresponds to $t = 0$, while $(2, 0, 2\pi)$ to $t = -2\pi$. Hence, the arc length between this points equals by symmetry to $s(2\pi) = 2\pi\sqrt{5}$.

3. Find the solution y of the differential equation $x^2y' + xy + 3 = 0$ with the initial condition $y(1) = 1$.

Solution. First we solve the homogeneous part of the DE:

$$x^2y' + xy = 0 \quad \Rightarrow \quad -\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad -\ln|y| = \ln|x| + k$$
$$\Rightarrow \quad y_h(x) = \frac{K}{x},$$

where $k, K \in \mathbb{R}$ are constants. Now we have to determine one particular solution. By variation of constants the form of the particular solution is

$$y_p(x) = \frac{K(x)}{x},$$

where $K(x)$ is a function of x. Thus,

$$y_p'(x) = \frac{K'(x)x - K(x)}{x^2} \tag{1}$$

and plugging (1) into the initial DE we get

$$x^2 \cdot \frac{K'(x)x - K(x)}{x^2} + x\frac{K(x)}{x} + 3 = 0.$$

Equivalently,

$$K'(x)x + 3 = 0. \tag{2}$$

We solve the DE (2) by separation of variables:

$$-\frac{dK}{3} = \frac{dx}{x} \quad \Rightarrow \quad -\frac{1}{3}K = \ln|x| \quad \Rightarrow \quad K = \ln\frac{1}{|x|^3}.$$

Since in the initial conditin $x > 0$, we have $K = \ln\frac{1}{x^3}$ and $y_p(x) = \ln\frac{1}{x^3} \cdot \frac{1}{x}$. So, the general solution of the DE is

$$y(x) = y_h(x) + y_p(x) = \left(K + \ln\frac{1}{x^3}\right)\frac{1}{x}.$$

The solution which passes through the point $(1, 1)$ is

$$y(1) = 1 = K + \ln 1 \quad \Rightarrow \quad K = 1 \quad \Rightarrow \quad y(x) = \left(1 + \ln\frac{1}{x^3}\right)\frac{1}{x}.$$

4. Solve the following system of differential equations:

$$x'(t) = -2x(t) + 5y(t),$$
$$y'(t) = x(t) + 2y(t),$$

with the initial conditions $x(0) = y(0) = 1$.

Solution. The matricial form of the system is the following:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

We compute the eigenvalues of A:

$$\det(A - \lambda I_2) = \det \begin{bmatrix} -2-\lambda & 5 \\ 1 & 2-\lambda \end{bmatrix} = (-2-\lambda)(2-\lambda) - 5$$
$$= \lambda^2 - 9 = (\lambda - 3)(\lambda + 3).$$

Thus the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$. The kernel of

$$A - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}$$

contains the vector

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The kernel of

$$A - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix}$$

contains the vector

$$u_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

So, the general solution of the system is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \cdot e^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cdot e^{-3t} \cdot \begin{bmatrix} -5 \\ 1 \end{bmatrix},$$

where C_1 and C_2 are constants. The solution, which satisfies $x(0) = y(0) = 1$, is:

$$C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad C_1 - 5C_2 = 1, \quad C_1 + C_2 = 1$$

$$\Rightarrow \quad C_1 = 1,\ C_2 = 0.$$