Mathematical modelling

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1. Given the system of equations $3x + y + z = 5$ in $x + y + z = 1$,

   (a) write down the matrix of the system,

   (b) find its Moore-Penrose inverse A^+,

   (c) find the point on the intersection of the planes $3x + y + z = 5$ in $x + y + z = 1$ closest to the origin.

   Solution :

   (a) The matrix of the system is

   $$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   (b) Since A is a $2x3$ rank matrix with rank 2, the Moore-Penrose pseudoinverse can be computed as

   $$A^+ = A^T(AA^T)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -1 & 3 \\ -1 & 3 \end{bmatrix}$$

   (c) The intersection of the planes closest to the origin can be computed simply as

   $$x = A^+b = \frac{1}{2}\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

   where

   $$b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

   is the right-hand side of the system of equations.

2. For the vector valued function

   $$f(u, v) = \begin{bmatrix} ue^v + \cos(\pi v) \\ u^2 \\ u - e^v \end{bmatrix}$$

(a) write down its Jacobian matrix,

(b) write down its linear approximation at the point $u = 1, v = 0$,

(c) use the linear approximation to compute the approximate value of $f(1.02, 0.01)$

Solution :

(a)

$$Jf(u,v) = \begin{bmatrix} e^v & ue^v - \pi\sin(\pi v) \\ 2u & 0 \\ 1 & -e^v \end{bmatrix}$$

(b) The linear approximation at the point $(1, 0)$ can be expressed as

$$\ell(x,y) = f(1,0) + Jf(1,0)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2+x+y \\ 1+2x \\ x-y \end{bmatrix}$$

(c) We compute

$$\ell(0.02, 0.01) = \begin{bmatrix} 2.03 \\ 1.04 \\ 0.01 \end{bmatrix}$$

3. Given the parametric curve $x(t) = t^2$, $y(t) = t^3 - 3t$

(a) find intersections with coordinate axes,

(b) show that there is a selfintersction at $(3, 0)$ and write down the two tangents to the curve at this point,

(c) find points with a horizontal or vertical tangent and sketch the curve.

(d) Is the curve smooth? Why or why not?

Solution :

(a) The equation $x(t) = 0$ has solutions $t_{1,2} = 0$ which gives the intersection with the y-coordinate axis at $A(0, 0)$. The equation $y(t) = 0$ has the solutions $t_1 = 0$ and $t_{2,3} = \pm\sqrt{3}$ which gives the intersections with the x-coordinate axis at $A(0, 0)$ and $B(3, 0)$.

(b) From the previous exercise we see the curve goes through the point $B(3,0)$ for two values of the parameter $t = \pm\sqrt{3}$ so the curve clearly has a selfintersection at this point. In parametric form the tangent lines can be expressed as

$$\begin{aligned} t_1(t) &= r(\sqrt{3}) + \dot{r}(\sqrt{3})t = \begin{bmatrix} 3+2\sqrt{3}t \\ 6t \end{bmatrix} \\ t_2(t) &= r(-\sqrt{3}) + \dot{r}(-\sqrt{3})t = \begin{bmatrix} 3-2\sqrt{3}t \\ 6t \end{bmatrix} \end{aligned}$$

(c) The curve is not smooth at the point $B(3,0)$ since its derivative at this point is not uniquely defined.

4. For the order 2 linear differential equation $\ddot{x} - 3\dot{x} + 2x = 0$

(a) write it in the form of a system of first order equations,

(b) find the general solution,

(c) find the solution satisfying the initial condition $x(0) = 0$, $\dot{x}(0) = 2$,

(d) classify $(0,0)$ as a critical point and sketch the phase portrait in the (x,v) plane.

Solution :

(a) By introducing the new variable $v = \dot{x}$ the equation can be written as the following first order system of equations:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 3v - 2x \end{aligned}$$

(b) The characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

which has solutions $\lambda_1 = 1$ and $\lambda_2 = 2$. The general solution has the form

$$x(t) = Ae^t + Be^{2t}$$

(c) Using the initial values we can find the values of the constants A and B in the general solution $A = -2$ and $B = 2$ and write

$$x(t) = -2e^t + 2e^{2t}$$

(d) The point $(0, 0)$ is unstable because the solution contains exponentials functions with positive coefficients in the exponent.