# Mathematical Modelling Exam

03. 06. 2022

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let

$$A = \begin{bmatrix} -2 & 2 \\ -3 & -2 \\ -2 & -3 \end{bmatrix}.$$

(a) Find the matrix $B \in \mathbb{R}^{3\times 2}$ of rank 1, which is the closest to $A$ in the Frobenius norm.

(b) Calculate $\|A - B\|_F$.

*Solution.* Let

$$A = U\Sigma V^T$$

be the truncated SVD of $A$, where

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \in \mathbb{R}^{3\times 2} \quad \text{with } U^T U = I_2,$$

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \in \mathbb{R}^{2\times 2} \quad \text{with } V^T V = I_2$$

and

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2), \quad \text{where } \sigma_1 \geq \sigma_2 \geq 0 \text{ are the singular values of } A.$$

By the Eckart-Young theorem it follows that

$$B = \sigma_1 u_1 v_1^T,$$

and hence

$$\|A - B\|_F = \|\sigma_2 u_2 v_2^T\|_F = \sigma_2 \|u_2 v_2^T\|_F.$$

We write $v_2 = \begin{bmatrix} v_{2,1} & v_{2,2} \end{bmatrix}^T$. Then

$$\begin{aligned} \|u_2 v_2^T\|_F &= \|[u_2 v_{2,1}, u_2 v_{2,2}]\|_F = \sqrt{\|u_2 v_{2,1}\|_F^2 + \|u_2 v_{2,2}\|_F^2} \\ &= \sqrt{(v_{2,1})^2 + (v_{2,2})^2} = \|v_2\|_F = 1, \end{aligned}$$

where in the third equality we used that $\|u_2\|_F = 1$. So we need to compute only $\sigma_1, \sigma_2, v_1, u_1$ to determine $B$ and $\|A - B\|_F$. We have that

$$A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix},$$

which implies

$$\det(A^T A - \lambda I_2) = (17 - \lambda)^2 - 8^2 = (17 - \lambda - 8)(17 - \lambda + 8) = (9 - \lambda)(25 - \lambda).$$

So the eigenvalues of $A^T A$ are $\lambda_1 = 25$, $\lambda_2 = 9$ and hence

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

The kernel of

$$A^T A - \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and hence

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now the first column of $U$ is

$$u_1 = \frac{1}{5} A v_1 = \frac{1}{5\sqrt{2}} \begin{bmatrix} 0 \\ -5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Finally,

$$B = 5 u_1 v_1^T = 5 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}$$

and

$$\|A - B\|_F = 3.$$

2. Let

$$\begin{aligned} x^2 + y &= 37, \\ x - y^2 &= 5, \\ x + y + z &= 3 \end{aligned}$$

be a nonlinear system and $v^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ a vector.

(a) Compute the approximation $v^{(1)}$ of the solution of the system using one step of Newton's method.

(b) Compute the tangent plane to the surface given by the equation $z = f(x, y)$, where

$$f(x, y) = 8xy + 4,$$

in the point $(1, 1)$.

*Solution.*

(a) We define a vector function of a vector variable $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x, y, z) = \begin{bmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{bmatrix} = \begin{bmatrix} x^2 + y - 37 \\ x - y^2 - 5 \\ x + y + z - 3 \end{bmatrix}.$$

We are searching for the solution of $F(x, y, z) = 0$ using Newton's method. We have that

$$v^{(1)} = v^{(0)} - \left(JF(v^{(0)})\right)^{-1} F(v^{(0)}),$$

where

$$JF(x,y,z) = \begin{bmatrix} \frac{\partial F_1(x,y,z)}{\partial x} & \frac{\partial F_1(x,y,z)}{\partial y} & \frac{\partial F_1(x,y,z)}{\partial z} \\ \frac{\partial F_2(x,y,z)}{\partial x} & \frac{\partial F_2(x,y,z)}{\partial y} & \frac{\partial F_2(x,y,z)}{\partial z} \\ \frac{\partial F_3(x,y,z)}{\partial x} & \frac{\partial F_3(x,y,z)}{\partial y} & \frac{\partial F_3(x,y,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 1 & 0 \\ 1 & -2y & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

is the Jacobian matrix of $F$. So

$$JF(v^{(0)}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We compute $\left(JF(v^{(0)})\right)^{-1}$ using Gaussian elimination:

$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]}_{\left[\ JF(v^{(0)})\ \middle|\ I_3\ \right]} \underbrace{\sim}_{\ell_1 \leftrightarrow \ell_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\sim}_{\ell_3 = \ell_3 - \ell_1 - \ell_2} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]}_{\left[\ I_3\ \middle|\ \left(JF(v^{(0)})\right)^{-1}\ \right]}.$$

So

$$v^{(1)} = -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -37 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 37 \\ -39 \end{bmatrix}.$$

(b) The surface is parametrized by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ 8xy+4 \end{bmatrix}.$$

The tangent plane to the surface in the point $(1,1)$ is

$$\begin{aligned} L(x,y) &= \begin{bmatrix} 1 \\ 1 \\ f(1,1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f(1,1)}{\partial x} \end{bmatrix}(x-1) + \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f(1,1)}{\partial y} \end{bmatrix}(y-1) \\ &= \begin{bmatrix} 1 \\ 1 \\ (8xy+4)(1,1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ (8y)(1,1) \end{bmatrix}(x-1) + \begin{bmatrix} 0 \\ 1 \\ (8x)(1,1) \end{bmatrix}(y-1) \\ &= \begin{bmatrix} 1 \\ 1 \\ 12 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}(x-1) + \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}(y-1) \\ &= \begin{bmatrix} x \\ y \\ -4+8x+8y \end{bmatrix} \end{aligned}$$

or $z = -4 + 8x + 8y$.

3. Let

$$f(t) = (\sin t, \cos(3t)), \quad t \in [0, 2\pi]$$

be the parametric curve.

(a) Find all points where the curve intersects the coordinate axes.

(b) Find all points on the curve where the tangent is horizontal or vertical.

(c) Sketch the curve.

*Solution.*

(a) The intersections with the $x$-axis correspond to $y(t) = 0$:

$$\cos(3t) = 0 \quad \Leftrightarrow \quad t \in \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}.$$

The corresponding points are

$$\begin{aligned} f\Big(\frac{\pi}{6}\Big) = f\Big(\frac{5\pi}{6}\Big) &= \Big(\sin\frac{\pi}{6}, 0\Big) = \Big(\frac{1}{2}, 0\Big), \\ f\Big(\frac{\pi}{2}\Big) &= \Big(\sin\frac{\pi}{2}, 0\Big) = (1, 0), \\ f\Big(\frac{7\pi}{6}\Big) = f\Big(\frac{11\pi}{6}\Big) &= \Big(\sin\frac{7\pi}{6}, 0\Big) = \Big(-\frac{1}{2}, 0\Big), \\ f\Big(\frac{3\pi}{2}\Big) &= \Big(\sin\frac{3\pi}{2}, 0\Big) = (-1, 0). \end{aligned}$$

The intersections with the $y$-axis correspond to $x(t) = 0$:

$$\sin t = 0 \quad \Leftrightarrow \quad t \in \{0, \pi, 2\pi\}.$$

The corresponding points are

$$\begin{aligned} f(0) = f(2\pi) &= (0, 1), \\ f(\pi) &= (0, -1). \end{aligned}$$

(b) The tangent to the curve is horizontal at the points where $y'(t) = 0$. Hence,

$$(\cos(3t))' = 0 \quad \Leftrightarrow \quad -3\sin(3t) = 0 \quad \Leftrightarrow \quad t \in \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi\right\}.$$

The corresponding points are

$$\begin{aligned} f(0) = f(2\pi) &= (0, 1), \\ f\Big(\frac{\pi}{3}\Big) &= \Big(\frac{\sqrt{3}}{2}, -1\Big), \\ f\Big(\frac{2\pi}{3}\Big) &= \Big(\frac{\sqrt{3}}{2}, 1\Big), \\ f(\pi) &= \Big(0, -1\Big), \\ f\Big(\frac{4\pi}{3}\Big) &= \Big(-\frac{\sqrt{3}}{2}, 1\Big), \\ f\Big(\frac{5\pi}{3}\Big) &= \Big(-\frac{\sqrt{3}}{2}, -1\Big). \end{aligned}$$

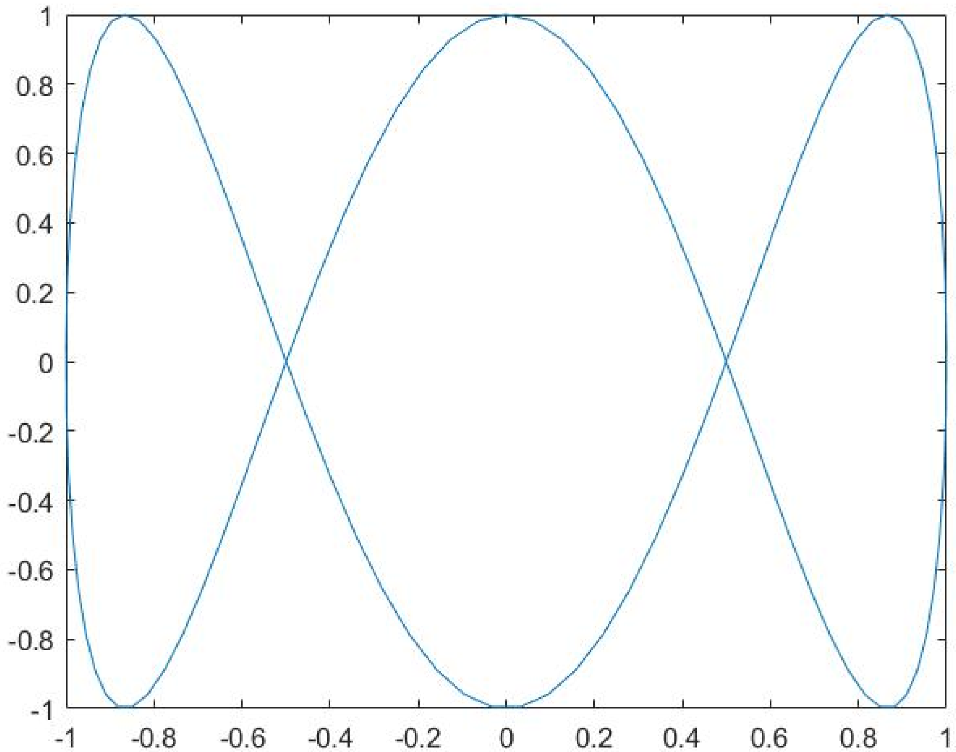
The tangent to the curve is vertical at the points where $x'(t) = 0$. Hence,

$$(\sin t)' = 0 \quad \Leftrightarrow \quad \cos t = 0 \quad \Leftrightarrow \quad t \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$

The corresponding points are

$$f\left(\frac{\pi}{2}\right) = (1, 0),$$
$$f\left(\frac{3\pi}{2}\right) = (-1, 0).$$

(c) The sketch of the curve is the following:




4. Consider the system of nonlinear differential equations

$$\dot{x} = x(3 - x - 2y),$$
$$\dot{y} = y(4 - 3x - y).$$

(a) Find the stationary points of the system.

(b) Compute the linearization of the system around the nontrivial stationary point, i.e., the one with both coordinates being nonzero.

(c) Solve the linear system from the previous question and sketch its phase portrait.

*Solution.*

(a) The stationary points satisfy $\dot{x} = \dot{y} = 0$. Hence:

$$x(3 - x - 2y) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 3 - 2y,$$
$$y(4 - 3x - y) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 4 - 3x.$$

If $x \neq 0$ and $y \neq 0$, then $y = 4 - 3(3 - 2y)$ and thus, $y = 1$ and $x = 1$. So the stationary points are:

$$(0, 0),\ (0, 4),\ (3, 0),\ (1, 1).$$

(b) The right side of the system is

$$\begin{aligned} f_1(x, y) &= 3x - x^2 - 2xy, \\ f_2(x, y) &= 4y - 3xy - y^2. \end{aligned}$$

The linearization of the system around $(1, 1)$ is

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} \approx \begin{bmatrix} f_1(1, 1) \\ f_2(1, 1) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(1,1)}{\partial x} & \frac{\partial f_1(1,1)}{\partial y} \\ \frac{\partial f_2(1,1)}{\partial x} & \frac{\partial f_2(1,1)}{\partial y} \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} \\ = \begin{bmatrix} (3 - 2x - 2y)(1, 1) & (-2x)(1, 1) \\ (-3y)(1, 1) & (4 - 3x - 2y)(1, 1) \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} \\ = \begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix}. \end{aligned}$$

(c) Introducing the new variables $X = x - 1$ and $Y = y - 1$, one obtains the following autonomous linear system:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

We have to compute the eigenvalues of $A$:

$$\begin{aligned} \det(A - \lambda I_2) = \det \begin{bmatrix} -1 - \lambda & -2 \\ -3 & -1 - \lambda \end{bmatrix} &= (-1 - \lambda)^2 - 6 \\ &= (\lambda + 1 - \sqrt{6})(\lambda + 1 + \sqrt{6}). \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = \sqrt{6} - 1$ and $\lambda_2 = -\sqrt{6} - 1$. The corresponding eigenspaces are:

$$\ker(A - \lambda_1 I_2) = \ker \begin{bmatrix} -\sqrt{6} & -2 \\ -3 & -\sqrt{6} \end{bmatrix} = \text{Lin}\left\{ \underbrace{\begin{bmatrix} 1 \\ -\frac{\sqrt{6}}{2} \end{bmatrix}}_{v_1} \right\}$$

and

$$\ker(A - \lambda_2 I_2) = \ker \begin{bmatrix} \sqrt{6} & -2 \\ -3 & \sqrt{6} \end{bmatrix} = \text{Lin}\left\{ \underbrace{\begin{bmatrix} 1 \\ \frac{\sqrt{6}}{2} \end{bmatrix}}_{v_2} \right\}.$$

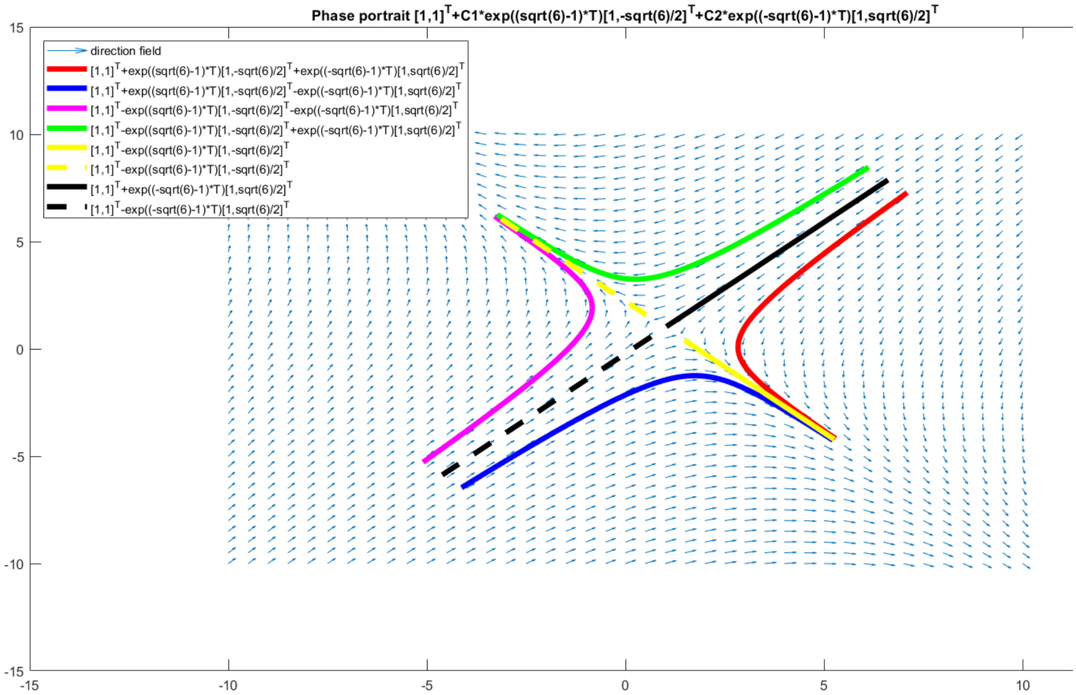
The solutions of the system are

$$C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

where $C_1, C_2 \in \mathbb{R}$. Hence, the solutions of the original system are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2,$$

where $C_1, C_2 \in \mathbb{R}$. The sketch of the phase portrait of the original system is the following:


Phase portrait [1,1]^T+C1*exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T+C2*exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
direction field
[1,1]^T+exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T+exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
[1,1]^T+exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T-exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
[1,1]^T-exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T-exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
[1,1]^T-exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T+exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
[1,1]^T-exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T
[1,1]^T-exp((sqrt(6)-1)*T)[1,-sqrt(6)/2]^T
[1,1]^T+exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
[1,1]^T-exp((-sqrt(6)-1)*T)[1,sqrt(6)/2]^T
15
10
5
0
-5
-10
-15
-15
-10
-5
0
5
10