Ime in priimek

Vpisna številka

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Osnove matematične analize: tretji računski izpit

23. avgust 2023

Čas pisanja je 90 minut. Dovoljena je uporaba 1 lista A4 formata s formulami. Uporaba kalkulatorja ali drugih pripomočkov ni dovoljena. Vse odgovore dobro utemelji!

1. naloga (25 točk)

a) (13 točk) Poišči vse kompleksne rešitve enačbe $z^2 + 2i\mathrm{Re}(z) = |z|$.

$z = x + iy \,;\; x, y \in \mathbb{R}$

$$(x+iy)^2 + 2ix = \sqrt{x^2+y^2}$$
$$x^2 + 2xyi - y^2 + 2xi = \sqrt{x^2+y^2}$$

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$$x^2 - y^2 = \sqrt{x^2+y^2}$$
$$2xy + 2x = 0$$
$$2x(y+1) = 0$$

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1. $\underline{x = 0}$

$$-y^2 = |y|$$
$$\underline{y = 0}$$

$z_1 = 0 + i0$

$\boxed{z_1 = 0}$

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2. $\underline{y = -1}$

$$x^2 - 1 = \sqrt{x^2+1} \quad /^2$$
$$x^4 - 2x^2 + 1 = x^2 + 1$$
$$x^4 - 3x^2 = 0$$
$$x^2(x^2-3) = 0$$

$x = 0$ //

$0^2 - (-1)^2 = \sqrt{0^2 + (-1)^2}$

$-1 = 1$ //

$\underline{x = \pm\sqrt{3}}$

$(\pm\sqrt{3})^2 - (-1)^2 = \sqrt{(\pm\sqrt{3})^2 + (-1)^2}$

$2 = 2$ ✓

$$\boxed{\begin{aligned} z_2 &= \sqrt{3} - i \\ z_3 &= -\sqrt{3} - i \end{aligned}}$$

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b) (12 točk) Poišči vse kompleksne rešitve enačbe $z^4 = -8 + 8\sqrt{3}i$.

$$a = -8 + 8\sqrt{3}i$$

$$|a| = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{64 + 364} = 16$$

$$\operatorname{tg}\varphi = \frac{8\sqrt{3}}{-8} = -\sqrt{3} \Rightarrow \varphi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

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$$z^4 = -8 + 8\sqrt{3}i$$

$$|z|^4 (\cos(4\alpha) + i\sin(4\alpha)) = 16 \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$|z|^4 = 16 \qquad 4\alpha = \frac{2\pi}{3} + 2k\pi$$

$$|z| = 2 \qquad \alpha = \frac{\pi}{6} + \frac{k\pi}{2}; \quad k \in \{0,1,2,3\}$$

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$$\boxed{z_k = 2\left(\cos\left(\frac{\pi}{6} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + \frac{k\pi}{2}\right)\right); \quad k \in \{0,1,2,3\}}$$

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2. naloga (25 točk)

Z uporabo korenskega, kvocientnega ali primerjalnega kriterija ugotovi, ali spodnji vrsti konvergirata ali divergirata.
Konvergentno vrsto tudi seštej. *(Namig: ugani formulo za n-to delno vsoto, tako da uporabiš razcep splošnega člena vrste na parcialne ulomke, ter vsoto vrste izračunaj po definiciji.)*

$$a) \sum_{n=1}^{\infty} \frac{n^n}{2^{2n}} \qquad b) \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^{2n}}$: KORENSKI KRITERIJ :

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^{2n}}} = \lim_{n\to\infty} \frac{n}{2^2} = \frac{1}{4} \lim_{n\to\infty} n = \infty$$

$\Rightarrow$ vrsta divergira 5

- $\displaystyle\sum_{n=1} \frac{1}{4n^2-1}$: PRIMERJALNI KRITERIJ :

Vemo : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira

$$\frac{1}{4n^2-1} \leq \frac{1}{n^2} \qquad \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$n^2 \leq 4n^2 - 1$$

$$3n^2 - 1 \geq 0 \checkmark$$

za $\forall n \geq 1$ $\qquad \Downarrow$

vrsta konvergira 5

$$a_n = \frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1)+B(2n-1)}{4n^2-1}$$

$$\oplus\left(\begin{array}{l} 2A+2B=0 \\ A-B=1 \,/\cdot 2 \\ 2A-2B=2 \end{array}\right.$$

$$4A=2 \Rightarrow A=\frac{1}{2} \Rightarrow B=-A=-\frac{1}{2}$$

$$a_n = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$ 5

$$S_1 = a_1 = \frac{1}{2\cdot 1} - \frac{1}{2\cdot 3}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} - \cancel{\frac{1}{2\cdot 3}} + \cancel{\frac{1}{2\cdot 3}} - \frac{1}{2\cdot 5}$$

$$\vdots$$

$$S_n = a_1 + a_2 + \ldots + a_n = \frac{1}{2} - \frac{1}{2(2n+1)}$$ 5

$$S = \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \underbrace{\frac{1}{2(2n+1)}}_{\to 0}\right) = \underline{\underline{\frac{1}{2}}}$$ 5

3. naloga (25 točk)

S ploskvijo $z = f(x,y)$, kjer je $f(x,y) = 2 - x - y$, ponazorimo relief neke pokrajine.

a) (5 točk) Izračunaj smerni odvod funkcije f v točki $(2,-1)$ v smeri najhitrejšega naraščanja.

$$f_x(x,y) = -1 \quad \to (\operatorname{grad} f)(x,y) = (-1,-1)$$

$$f_y(x,y) = -1 \quad (\operatorname{grad} f)(2,-1) = (-1,-1) \leftarrow$$ smer najhitrejšega naraščanja

$$f_{(-1,-1)}(2,-1) = \frac{(-1,-1)\cdot(-1,-1)}{\|(-1,-1)\|} = \frac{(-1)^2+(-1)^2}{\sqrt{(-1)^2+(-1)^2}} = \frac{2}{\sqrt{2}} = \underline{\underline{\sqrt{2}}}$$ 5

b) (20 točk) Poišči najvišje in najnižje ležečo točko na poti, katere projekcija dane pokrajine na (x, y) ravnino ima enačbo $x^2 + xy + y^2 = 2$. *(Namig: Vezani ekstremi.)*

$$F(x,y,\lambda) = f(x,y) - \lambda g(x,y) \qquad g(x,y) = x^2+xy+y^2-2$$

$$F(x,y,\lambda) = 2-x-y-\lambda(x^2+xy+y^2-2) \quad 5$$

$$F_x(x,y,\lambda) = -1-2\lambda x-\lambda y = 0 \rightarrow \lambda(2x+y) = -1 \rightarrow \lambda = \frac{-1}{2x+y}$$

$$F_y(x,y,\lambda) = -1-\lambda x-2\lambda y = 0 \rightarrow \lambda(2y+x) = -1 \rightarrow \lambda = \frac{-1}{2y+x}$$

$$F_\lambda(x,y,\lambda) = -(x^2+xy+y^2-2) = 0$$

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$$\lambda = \lambda$$

$$-\frac{1}{2x+y} = -\frac{1}{2y+x}$$

$$2y+x = 2x+y$$

$$-x+y = 0$$

$$\boxed{x = y}$$

$$x^2+x^2+x^2-2 = 0$$

$$3x^2 = 2$$

$$x_{1,2} = \pm\sqrt{\frac{2}{3}}$$

$$y_{1,2} = \pm\sqrt{\frac{2}{3}} \quad 5$$

$$\boxed{\begin{array}{ll} T_1\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, 2-2\sqrt{\frac{2}{3}}\right) & \text{MIN} \\ T_2\left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, 2+2\sqrt{\frac{2}{3}}\right) & \text{MAX} \end{array}} \quad 5$$

4. naloga (25 točk)

Dana je funkcija $f(x) = (1 + 2x)e^{-x}$.

a) (7 točk) Izračunaj $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}(1+2x)e^{-x} = \lim_{x\to\infty}\frac{1+2x}{e^x} \overset{L'H}{=} \lim_{x\to\infty}\frac{2}{e^x} = \underline{\underline{0}} \quad 7$$

b) (12 točk) Ali obstaja $\int_0^\infty f(x)dx$? Če obstaja, ga izračunaj, sicer utemelji, zakaj ne obstaja.

$$\int_0^\infty (1+2x)e^{-x}\,dx = -(1+2x)e^{-x}\Big|_0^\infty + 2\int_0^\infty e^{-x}\,dx =$$

$$u = 1+2x \rightarrow du = 2\,dx$$
$$dv = e^{-x}dx \rightarrow v = -e^{-x}$$

3

$$= \lim_{x\to\infty} -(1+2x)e^{-x} + 1\cdot e^{0} - 2e^{-x}\Big|_0^\infty =$$

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$$= 0 + 1 - 2\Big(\lim_{x\to\infty} \underset{0}{\underset{\parallel}{e^{-x}}} - \underset{1}{\underset{\parallel}{e^{0}}}\Big) = 1 - 2(-1) = \underline{\underline{3}}$$

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c) (6 točk) Izračunaj volumen vrtenine, ki jo dobimo, če graf funkcije

$$h(x) = \sqrt{f(x)}$$

zavrtimo okrog x-osi na intervalu $[0, \infty)$.

$$V = \pi \int_0^\infty h^2(x)\,dx = \pi \int_0^\infty f(x)\,dx = \underline{\underline{3\pi}}$$

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