Osnove matematične analize

Vaje, 2. teden

baza indukcije

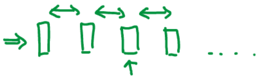
Če trditev 1. velja za nek $n_0 \in \mathbb{N}$ in

2. če velja za n, potem velja tudi za $n+1$,

potem ta trditev velja za vse $n \geq n_0$.

indukcijski korak

indukcijska predpostavka



1. * Z matematično indukcijo dokaži:

(a) $1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$,

- baza $n=1$

$$1 \cdot 2 \overset{?}{=} \frac{1}{3} \cdot 1 \cdot 2 \cdot 3$$
$$2 = 2 \ ✓$$

- indukcijski korak

velja za n: $1 \cdot 2 + \cdots + n(n+1) \overset{\text{I.P.}}{=} \frac{1}{3}n(n+1)(n+2)$ (to vemo, ind. pred.)

radi bi videli (RBV), da velja za $n+1$:

$$\underset{L=}{1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2)} \overset{?}{=} \underset{D}{\frac{1}{3}(n+1)(n+2)(n+3)}$$

$$L = 1 \cdot 2 + \cdots + n(n+1) + (n+1)(n+2) \overset{\text{I.P.}}{=} \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2) = (n+1)(n+2)\Big(\underbrace{\overset{\frac{1}{3}\cdot 3}{\frac{1}{3}n + 1}}_{\frac{1}{3}(n+3)}\Big) =$$

$$= \frac{1}{3}(n+1)(n+2)(n+3) = D$$

$L = D$ ✓

(b) $1^3 + 2^3 + \ldots + n^3 = n^2(n+1)^2/4$,

- baza: $n=1$

$$1^3 = 1^2 \cdot 2^2 / 4$$
$$1 = 1$$

$L = D$ ✓ za $n=1$ trditev drži

- ind. korak $n \longrightarrow n+1$

velja za n (I.P.): $1^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

RBV, da velja za $n+1$: $\underset{L=}{1^3 + \cdots + n^3 + (n+1)^3} \overset{?}{=} \underset{D}{\frac{(n+1)^2(n+2)^2}{4}}$

$$L = 1^3 + \cdots + n^3 + (n+1)^3 \overset{\text{I.P.}}{=} \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + n + 1\right) = (n+1)^2 \frac{n^2 + 4(n+1)}{4} =$$

$$= \frac{1}{4}(n+1)^2\Big(\underbrace{n^2 + 4n + 4}_{(n+2)^2}\Big) = \frac{1}{4}(n+1)^2(n+2)^2 = D$$

$L = D$ če velja za n, velja za $n+1$

(c) $n! > 2^{n-1}$ za $n > 2$,

$n=0$ $\quad 0! > 2^{-1}$ $\quad 1 > \frac{1}{2}$ ✓

$n=1$ $\quad 1! > 2^{0}$ $\quad 1 > 1$ //

$n=2$ $\quad 2! > 2^{1}$ $\quad 2 > 2$ //

- **baza**: $n=3$

$$3! > 2^{3-1} \ ?$$

$$1\cdot 2\cdot 3 > 2^2 \ ?$$

$$6 > 4 \ \checkmark$$

$$\boxed{n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)n}$$

- **ind. korak** $n \to n+1$

vemo: $n! > 2^{n-1}$ I.P.

RBV: $\underbrace{(n+1)!}_{L} > \underbrace{2^n}_{D}$?

$n+1 > 2$, ker $n \geq 3$

$n > 2$ $\quad n+1 > 3$

$n+1 > 2$

$$L = (n+1)! = \underbrace{1\cdot 2\cdot 3\cdot \ldots \cdot n}_{n!}(n+1) = n!(n+1) > 2^{n-1}(n+1) > 2^{n-1}\cdot 2 = 2^n = D$$

$L=D$ ✓

$$\boxed{\begin{array}{l} x_1 > \bullet > \bullet = \bullet = \bullet > \bullet > x_k \longrightarrow x_1 > x_k \\ x_1 > \bullet = \bullet = \bullet < \bullet = \bullet = x_k \longrightarrow ??? \end{array}}$$

(d) $1\cdot 4 + 2\cdot 4^2 + 3\cdot 4^3 + \cdots + n\cdot 4^n > \frac{(3n-1)4^{n+1}}{9}$.

baza: $n=1$

$$1\cdot 4 > \frac{(3-1)\cdot 4^2}{9}$$

$$4 > \frac{2\cdot 16}{9}$$

$$4 > \frac{32}{9} = \frac{27}{9} + \frac{5}{9} = 3{,}\ldots \ \checkmark$$

$n \to n+1$ $\quad 3(n+1)-1 = 3n+3-1 = 3n+2$

$(n+1)+1 = n+2$

ind. korak: $n \to n+1$

vemo: $1\cdot 4 + 2\cdot 4^2 + \ldots + n\cdot 4^n > \frac{(3n-1)\cdot 4^{n+1}}{9}$

RBV: $\underbrace{1\cdot 4 + \cdots + n\cdot 4^n + (n+1)\cdot 4^{n+1}}_{L} > \underbrace{\frac{(3n+2)\cdot 4^{n+2}}{9}}_{D}$???

$$L = 1\cdot 4 + \cdots + n\cdot 4^n + (n+1)\cdot 4^{n+1} > \frac{(3n-1)\cdot 4^{n+1}}{9} + (n+1)\cdot 4^{n+1} = 4^{n+1}\left(\frac{3n-1}{9} + \underbrace{n+1}_{\frac{9(n+1)}{9}}\right) =$$

$$= \frac{4^{n+1}}{9}\left(\underbrace{3n-1+9(n+1)}_{3n-1+9n+9}\right) = \frac{4^{n+1}}{9}\underbrace{(12n+8)}_{4(3n+2)} = \frac{4^{n+2}}{9}(3n+2) = \frac{(3n+2)\cdot 4^{n+2}}{9} = D$$

$L > D$ ✓

2. * Z uporabo matematične indukcije utemelji, da za vsako naravno število $n \geq 2$ velja:

$$\log\left(1-\frac{1}{2^2}\right)+\log\left(1-\frac{1}{3^2}\right)+\cdots+\log\left(1-\frac{1}{n^2}\right)=\log\left(\frac{n+1}{2n}\right).$$

$$\boxed{\log(a)+\log(b)=\log(ab)}$$

$$\log\left(\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot\ \ldots\ \cdot\left(1-\frac{1}{n^2}\right)\right)=\log\left(\frac{n+1}{2n}\right)$$

$$\boxed{\log a=\log b \Rightarrow a=b}$$

$$\boxed{\underbrace{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}}_{\bigstar}\quad \text{za } n\geq 2}$$

$n \to n+1$

$2(n+1)=2n+2$

- baza $n=2$

$$\left(1-\frac{1}{2^2}\right)=\frac{2+1}{2\cdot 2}\ ?$$

$$1-\frac{1}{4}=\frac{3}{4}\ ?$$

$$\frac{3}{4}=\frac{3}{4}\ \checkmark$$

- ind. korak $n\to n+1$

vemo: $\bigstar$

RBV: $\underset{L}{\underbrace{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)}}=\underset{D}{\frac{n+2}{2n+2}}\ ?$

$$L=\left(1-\frac{1}{2^2}\right)\ldots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)\overset{\bigstar\ \text{I.P.}}{=}\frac{n+1}{2n}\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+1}{2n}\cdot\frac{(n+1)^2-1}{(n+1)^2}=$$

$$=\frac{(n+1)^2-1}{2n(n+1)}=\frac{n^2+2n+\cancel{1}-\cancel{1}}{2n(n+1)}=\frac{\cancel{n}(n+2)}{2\cancel{n}(n+1)}=\frac{n+2}{2(n+1)}=\frac{n+2}{2n+2}=D$$

$L=D\ \checkmark$

3. Dokaži, da je za vsako naravno število $n > 0$ število $11^{n+1} + 12^{2n-1}$ deljivo s 133.

$11^2 + 12 = 121 + 12 = 133$

$12^2 - 11 = 144 - 11 = 133$

4. * Dokaži, da je za vsako naravno število n število $7^{n+2}+8^{2n+1}$ deljivo s 57.

57 deli $7^{n+2}+8^{2n+1}$ za vse n

- baza $n=1$

57 deli $7^3+8^3 = 343+512 = \underset{57\cdot 15}{\overset{\shortparallel}{855}}$?

57 deli $57\cdot 15$? ✓

$n=0$

$7^2+8^1 = 49+8 = 57$

57 deli 57 ✓

$$\boxed{\begin{aligned} a \text{ deli } b &\Leftrightarrow b = a\cdot k \text{ za nek } k \in \mathbb{Z} \\ &\Leftrightarrow b \text{ je deljiv z } a \end{aligned}}$$

- ind. korak : $n \rightarrow n+1$

vemo : 57 deli $7^{n+2}+8^{2n+1}$ (vemo : $7^{n+2}+8^{2n+1} = 57\cdot k$ za nek $k\in\mathbb{Z}$)

RBV : 57 deli $7^{n+3}+8^{2n+3}$? (RBV $7^{n+3}+8^{2n+3} = 57\cdot l$ za nek $l\in\mathbb{Z}$)

$$7^{n+3}+8^{2n+3} = 7\cdot 7^{n+2}+8^{2n+3} = 7\cdot 7^{n+2} + \overbrace{7\cdot 8^{2n+1} - 7\cdot 8^{2n+1}}^{0} + 8^{2n+3} =$$

$$= 7\big(\underbrace{7^{n+2}+8^{2n+1}}_{57k}\big) - 7\cdot 8^{2n+1} + 8^{2n+3} = 7\cdot 57k + 8^{2n+1}\big(\overbrace{-7+8^2}^{64-7=57}\big) =$$

$$= 7\cdot 57k + 8^{2n+1}\cdot 57 = 57\big(\underbrace{7k+8^{2n+1}}_{l\in\mathbb{Z}}\big) = 57l \rightarrow 57 \text{ deli } 7^{n+3}+8^{2n+3} \ ✓$$

$7^2+8^1 = 57$

$8^2-7^1 = 57$

6. * Za vsako od naslednjih množic določi infimum in supremum. Ali obstaja minimum ali maksimum?

inf A = infimum = največja spodnja meja = največje število, ki je $\leq$ od vseh iz A

sup A = supremum = najmanjša zgornja meja = najmanjše ... $\geq$ ...

min A = minimum = najmanjši element v A (če obstaja)

max A = maksimum = največji -||- (-||-)

$(a, b]$

$a \notin (a, b]$

$a + \varepsilon \in (a, b]$

$a + \frac{\varepsilon}{2} \in (a, b]$

$a + \frac{\varepsilon}{4} \in (a, b]$

$\vdots$

$A = (a, b]$ inf A = a min A ne obstaja

sup A = b max A = b

(a) $A = \{x \in \mathbb{R} ;\ \underline{||x-1|-2| \geq 1}\}$,

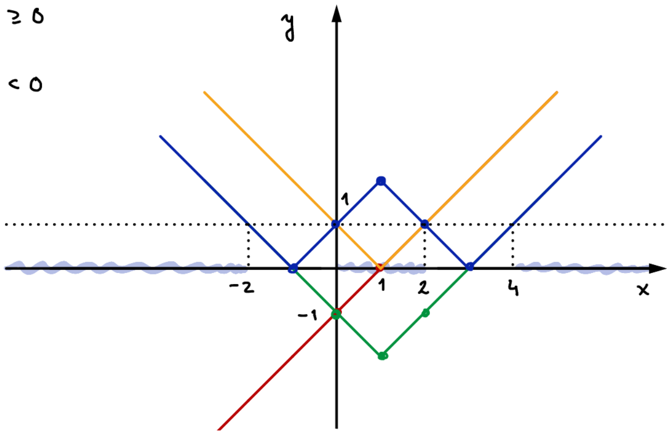
$$|x| = \begin{cases} x ; & x \geq 0 \\ -x ; & x < 0 \end{cases}$$

1. način $||x-1|-2| \geq 1$

$$||x-1|-2| = \begin{cases} |x-1|-2 ; & |x-1|-2 \geq 0 \\ 2-|x-1| ; & |x-1|-2 < 0 \end{cases}$$

2. način: narišemo graf $f(x) = ||x-1|-2|$

- $y = x-1$
- $y = |x-1|$
- $y = |x-1|-2$
- $y = ||x-1|-2|$



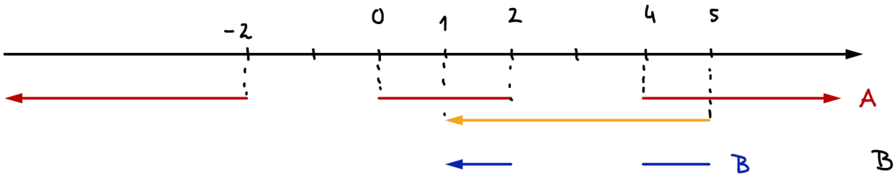
$A = (-\infty, -2] \cup [0, 2] \cup [4, \infty)$

min A in max A ne obstajata

inf A -||- sup A -||- (ali inf A $= -\infty$, sup A $= \infty$)

(b) $B = \{x \in \mathbb{R} ;\ \overbrace{||x-1|-2| \geq 1}^{x \in A}, \overbrace{x \leq 5 \text{ in } x > 1}^{x \in (1,5]}\}$,

$B = A \cap (1, 5]$



$B = (1, 2] \cup [4, 5]$

inf B = 1 sup B = 5

min B = ne obstaja max B = 5

(c) $C = \{2 + \sin x ;\ x \in \mathbb{R}\}$,

(d) $D = \{x \in \mathbb{R} ;\ \log 2 + \log (x^2 - 1) \leq 2 \log |x - 1|\}$,

(e) $E = \{x \in \mathbb{R} ;\ \log 2 + \log |x^2 - 1| \leq 2 \log |x - 1|\}$,

(f) $F = \{x \in \mathbb{R} ;\ \log 2 + \log (x^2 - 1) \leq 2 \log (x - 1)\}$.