1. Denoting $\mathbf{x} = [x, y]^\mathsf{T}$ find the general solutions to the system of differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ in case A is the following matrix:

   (a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,   (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,   (c) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   Use Octave to draw phase portraits (solution trajectories for several initial values) for each of the systems above. How do eigenvalues of the matrix A affect the behaviour of the solutions?

2. The basic SIR compartmental model for modelling epidemics divides the population of size N into 3 compartments: susceptibles S, infectious I, and recovered R. The dynamics of this model is governed by the system of differential equations

$$\dot{S} = -\frac{\beta I S}{N},$$
$$\dot{I} = \frac{\beta I S}{N} - \gamma I,$$
$$\dot{R} = \gamma I,$$

   where $S(t)$, $I(t)$, and $R(t)$ are functions of time, while $\dot{S} = \frac{dS}{dt}$, $\dot{I} = \frac{dI}{dt}$, and $\dot{R} = \frac{dR}{dt}$ are their derivatives. (The quotient $\frac{\beta}{\gamma}$ is often denoted by R_0 – basic reproduction number.)

   (a) Confirm that $\dot{S} + \dot{I} + \dot{R} = 0$, hence $S(t) + I(t) + R(t) = N$ (a constant – the size of population N).

   In what follows we assume $N = 1$, the functions S, I, and R therefore represent respective *fractions* of the population.

   (b) Show that $R(t) = 1 - S(t) - I(t)$, hence (w.r.t. solving the problem) only the first two equations of the above system are needed.

   (c) Use rk4 and find the solutions of the system above for some chosen initial conditions and some chosen values of parameters β and γ.

3. *A mathematical pendulum* is a point mass m suspended on a massless (and inflexible) rod of length ℓ attached to a frictionless pivot. The mass m is acted upon by gravity mg, the displacement angle at time t is denoted by $\phi(t)$.

   (a) Show that ϕ solves the differential equation

$$\ddot{\phi} + \frac{g}{\ell} \sin(\phi) = 0.$$

   (b) Substitute $\omega = \dot{\phi}$ to convert the equation of order 2 above into a system of two order 1 equations.

   (c) Plot the phase diagram $(\phi, \dot{\phi}) = (\phi, \omega)$ of the equation above. Use rk4.