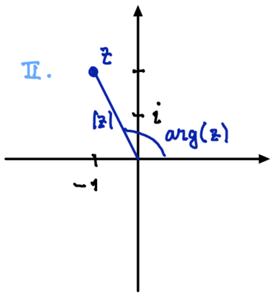
Osnove matematične analize
Vaje 2

1. Število $z = \frac{1+3i}{1-i}$ zapiši v obliki $x + iy$ in izračunaj $|z|$ ter $\arg(z)$.

Rešitev: $z = -1 + 2i$, $|z| = \sqrt{5}$, $\arg(z) = 2.03$.

$$z = \frac{1+3i}{1-i} = \frac{(1+3i)(1+i)}{(1-i)(1+i)} = \frac{1+3i+i-3}{1-(-1)} = \frac{-2+4i}{2} = \underline{\underline{-1+2i}}$$



$$|z| = \sqrt{(-1)^2 + 2^2} = \underline{\underline{\sqrt{5}}}$$

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$\arg(z) = \operatorname{arctg}\left(\frac{y}{x}\right)$

atan2(2, −1) vrne rezultat v pravem kvadrantu

$$\arg(z) = \operatorname{arctg}\left(\frac{2}{-1}\right) = \operatorname{arctg}(-2) = \begin{cases} -63.4^\circ \underbrace{+ 180^\circ}_{\text{II.}} = \underline{\underline{116.6^\circ}} \\ -1.11 \underbrace{+ \pi}_{\text{II.}} = \underline{\underline{2.03}} \text{ rad} \end{cases}$$

2. * Nariši množico točk:

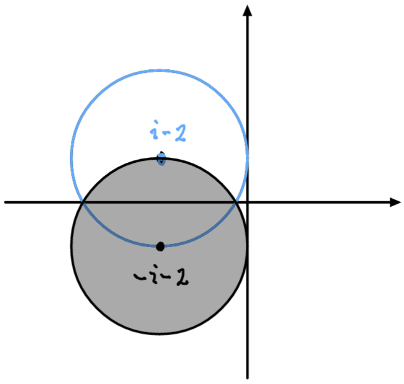
(a) $|\overline{z} + 2 - i| \leq 2$;

(b) $\mathrm{Re}(\overline{z} + 2 - i) \leq 2$;

(c) $\mathrm{Im}(\overline{z} + 2 - i) \leq 2$.

Rešitve: prva množica je krog z radijem 2 in središčem $(-2, -1)$, druga je polravnina $\{x \leq 0\}$, tretja je polravnina $\{y \geq -3\}$.

a) $A = \{ z \in \mathbb{C} \; ; \; \underbrace{d}_{\text{razdalja}}(\overline{z}, i-2) \leq 2 \} = \underbrace{\{ \underbrace{\overline{z}}_{\text{zrcaljeno čez x-os}} \in \mathbb{C} \; ; \; \underbrace{d(z, i-2) \leq 2}_{\text{krog z radijem 2 in središčem } i-2} \}}_{\text{krog z radijem 2 in središčem } -i-2}$



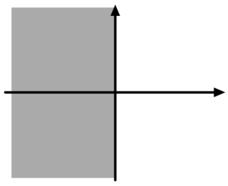
2. način:

$|\overline{z} + 2 - i| = |x - iy + 2 - i| = |(x+2) - i(y+1)| = \sqrt{(x+2)^2 + (y+1)^2} \leq 2$

$\Rightarrow \underbrace{(x+2)^2 + (y+1)^2 \leq 2^2}_{\text{krog z radijem 2 in središčem } (-2,-1)}$

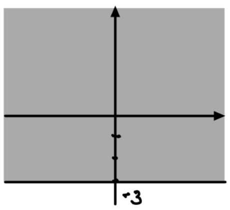
b) $\mathrm{Re}(\overline{z} + 2 - i) = \mathrm{Re}\big((x+2) - i(y+1)\big) = x + 2 \leq 2$

$\underline{\underline{x \leq 0}}$



c) $\mathrm{Im}(\overline{z} + 2 - i) = \mathrm{Im}\big((x+2) - i(y+1)\big) = -y - 1 \leq 2$

$\underline{\underline{y \geq -3}}$



3. * Reši naslednje enačbe

(a) $z^2 + z = 1$

(b) $(2+i)z + 2z - 3 = 4 + 6i$

Rešitve: (a) $z_1 = \frac{-1+\sqrt{5}}{2}$, $z_2 = \frac{-1-\sqrt{5}}{2}$, (b) $z = 2 + i$.

a) $z^2 + z = 1$

1. način: $z = x + iy$

$$(x+iy)^2 + (x+iy) = 1$$

$$x^2 + 2ixy - y^2 + x + iy = 1$$

$$\underbrace{(x^2 + x - 1 - y^2)}_{0} + i\underbrace{(2xy + y)}_{0} = 0 = 0 + i \cdot 0$$

I: $x^2 + x - 1 - y^2 = 0$

II: $2xy + y = 0 \longrightarrow y(2x+1) = 0$

$y = 0$:

I: $x^2 + x - 1 = 0$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} \quad y_{1,2} = 0$$

$$z_1 = \frac{-1+\sqrt{5}}{2} + 0 \cdot i$$

$$z_2 = \frac{-1-\sqrt{5}}{2} + 0 \cdot i$$

$x = -\frac{1}{2}$:

I: $\frac{1}{4} - \frac{1}{2} - 1 = y^2$

$y^2 = -\frac{5}{4}$ // $y^2 \geq 0$ za vse $y \in \mathbb{R}$

2. način: $z^2 + z = 1$

$$z^2 + z - 1 = 0$$

$$z_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$z_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

(Ker imamo polinom druge stopnje z $\mathbb{R}$ koeficienti, vemo, da ima dve ničli v $\mathbb{C}$ in dve smo našli.)

b) $z = x + iy$

$$(2+i)(x+iy) + 2(x+iy) - 3 = 4 + 6i$$

$$2x + 2iy + ix - y + 2x + 2iy - 3 = 4 + 6i$$

Re: $4x - y - 3 = 4 \longrightarrow 4(6-4y) - y - 3 = 4$

Im: $4y + x = 6 \longrightarrow x = 6 - 4y$

$$24 - 16y - y = 7$$

$$17y = 17$$

$$y = 1$$

$$x = 2$$

$$z = 2 + i$$

4. * Reši naslednje enačbe:

(a) $2z^2 - 3\overline{z}^2 = 10i$;

(b) $\overline{z} - iz^2 = 0$.

Rešitve: (a) $z_1 = 1 + i$, $z_2 = -1 - i$, (b) $z_1 = 0$, $z_2 = i$, $z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, $z_4 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$.

a) $z = x + iy$

$$2(x+iy)^2 - 3(x-iy)^2 = 10i$$

$$2(x^2 + 2ixy - y^2) - 3(x^2 - 2ixy - y^2) = 10i$$

$$-x^2 + 10ixy + y^2 = 10i + 0$$

Re: $y^2 - x^2 = 0 \longrightarrow y^2 = x^2 \longrightarrow \left(\frac{1}{x}\right)^2 = x^2 \longrightarrow \frac{1}{x^2} = x^2$

$x^4 = 1$

$\downarrow$

4 rešitve v $\mathbb{C}$ $(\pm 1, \pm i)$

$x \in \mathbb{R}$ $(x = \mathrm{Re}(z)) \Rightarrow \underline{\underline{x = \pm 1}}$

Im: $10xy = 10 \longrightarrow xy = 1 \longrightarrow y = \frac{1}{x}$

$x = 1 \longrightarrow y = 1 \longrightarrow \underline{\underline{z = 1 + i}}$

$x = -1 \longrightarrow y = -1 \longrightarrow \underline{\underline{z = -1 - i}}$

b) $z = x + iy$

$$x - iy - i(x+iy)^2 = 0$$

$$x - iy - i(x^2 + 2ixy - y^2) = 0$$

$$x - iy - ix^2 + 2xy + iy^2 = 0$$

Re: $x + 2xy = 0 \longrightarrow x(1 + 2y) = 0$

Im: $-x^2 - y + y^2 = 0$

$x = 0$:

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0$: $\underline{\underline{z_1 = 0}}$

$y = 1$: $\underline{\underline{z_2 = i}}$

$y = -\frac{1}{2}$:

$-x^2 + \frac{1}{2} + \frac{1}{4} = 0$

$x^2 = \frac{3}{4}$

$x = \pm \frac{\sqrt{3}}{2}$

$\underline{\underline{z_3 = \frac{\sqrt{3}}{2} - \frac{1}{2}i}}$

$\underline{\underline{z_4 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i}}$

5. Z uporabo polarne oblike in de Moivrove formule izračunaj

(a) * $\left(-\frac{1}{2}+\frac{i}{2}\right)^8$,

(b) * $\left(-1-i\sqrt{3}\right)^{20}$,

(c) $(1-i)^{5000}$,

(d) $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$.

Rešitve: (a) $\frac{1}{16}$, (b) $2^{19}(-1+i\sqrt{3})$, (c) 2^{2500}, (d) $2^9(1-i\sqrt{3})$.

$$\boxed{(\cos(x)+i\sin(x))^n = \cos(nx)+i\sin(nx)}$$

Eulerjeva formula: $e^{ix} = \cos x + i\sin x$

$$(e^{ix})^n = e^{inx} = \cos(nx)+i\sin(nx)$$

$$\boxed{\text{Polarni zapis: } z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi} \qquad |z| = r = \sqrt{Re(z)^2+Im(z)^2} \qquad \varphi = \arg(z) = \operatorname{arctg}\left(\frac{Im(z)}{Re(z)}\right)}$$

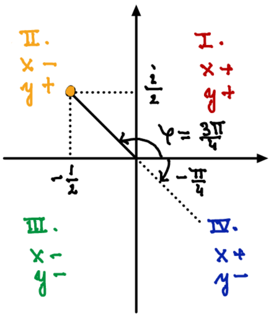
a) $z = -\frac{1}{2} + i\cdot\frac{1}{2}$

$$|z| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\varphi = \arg(z) = \operatorname{arctg}\left(\frac{Im(z)}{Re(z)}\right) = \operatorname{arctg}\left(\frac{\frac{1}{2}}{-\frac{1}{2}}\right) = $$

$$= \operatorname{arctg}(-1) = -\frac{\pi}{4} + \_\_\cdot\pi$$

atan2$\left(\frac{1}{2},-\frac{1}{2}\right)$ = atan2$(1,-1)$
vrne kot v pravem kvadrantu
(C, Java, Python, ...)



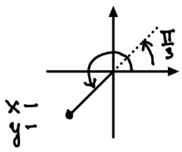
$$\longrightarrow \quad \varphi = -\frac{\pi}{4}+\pi = \frac{3\pi}{4} \quad (135°)$$

$$z^8 = \left(\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right)^8 = \frac{1}{2^4}\left(\cos\underbrace{\frac{24\pi}{4}}_{\substack{6\pi \\ 2\pi k = 0}}+i\sin\frac{24\pi}{4}\right) = \frac{1}{16}\left(\underbrace{\cos 0}_{1}+i\underbrace{\sin 0}_{0}\right) = \frac{1}{16}$$

b) $z = -1-i\sqrt{3}$

$$|z| = \sqrt{1+3} = 2$$

$$\varphi = \operatorname{arctg}\left(\frac{-\sqrt{3}}{-1}\right) = \operatorname{arctg}(\sqrt{3}) = \frac{\pi}{3}+\pi = \frac{4\pi}{3}$$



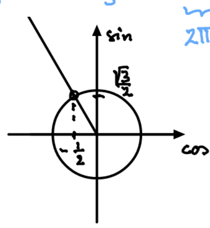
$$z^{20} = 2^{20}\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)^{20} =$$

$$= 2^{20}\left(\cos\frac{80\pi}{3}+i\sin\frac{80\pi}{3}\right) =$$

$$\frac{80\pi}{3} = \frac{(78+2)\pi}{3} = \underbrace{26\pi}_{2\pi k=0} + \overbrace{\frac{2\pi}{3}}^{120°}$$

$$= 2^{20}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) =$$

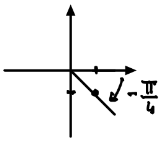
$$= 2^{20}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2^{19}(-1+i\sqrt{3})$$



c) $z = 1-i$ 5000

$$|z| = \sqrt{2}$$

$$\varphi = \operatorname{arctg}(-1) = -\frac{\pi}{4}$$



$$z^{5000} = (\sqrt{2})^{5000}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)^{5000} =$$

$$= 2^{2500}\left(\cos(\underbrace{1250\pi}_{0})+i\sin(1250\pi)\right) = 2^{2500}\left(\underbrace{\cos 0}_{1}+i\underbrace{\sin 0}_{0}\right) =$$

$$= 2^{2500}$$

d) $z = \frac{1+i\sqrt{3}}{1-i} = \frac{(1+i\sqrt{3})(1+i)}{(1-i)(1+i)} = \frac{1+i+i\sqrt{3}-\sqrt{3}}{2} = \underbrace{\frac{1-\sqrt{3}}{2}}_{-} + i\underbrace{\frac{1+\sqrt{3}}{2}}_{+}$

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$$|z| = \sqrt{\left(\frac{1-\sqrt{3}}{2}\right)^2+\left(\frac{1+\sqrt{3}}{2}\right)^2} = \frac{1}{2}\sqrt{\underbrace{(1-\sqrt{3})^2+(1+\sqrt{3})^2}_{1-2\sqrt{3}+3+1+2\sqrt{3}+3}} = \frac{1}{2}\sqrt{8} = \frac{1}{2}2\sqrt{2} = \sqrt{2}$$

$$\varphi = \operatorname{arctg}\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right) \begin{matrix} /\cdot(1+\sqrt{3}) \\ /\cdot(1+\sqrt{3}) \end{matrix} = \operatorname{arctg}\left(\frac{(1+\sqrt{3})^2}{-2}\right) = \operatorname{arctg}\left(\frac{4+2\sqrt{3}}{-2}\right) = \operatorname{arctg}(-2-\sqrt{3}) = -\frac{5\pi}{12}+\pi = \frac{7\pi}{12} \xrightarrow{\cdot 20} \frac{140\pi}{12} = \underbrace{10\pi}_{0}+\frac{20\pi}{12} = \frac{5\pi}{3}$$

dober kalkulator

$$z^{20} = (\sqrt{2})^{20}\left(\cos\underbrace{\frac{5\pi}{3}}_{-60°}+i\sin\frac{5\pi}{3}\right) = 2^{10}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = 2^9(1-i\sqrt{3})$$

6. * Nariši naslednjo podmnožico v $\mathbb{C}$:

polarni kot med 0 in $\frac{\pi}{4}$

$$A = \{z \in \mathbb{C}\,;\ 1 < |z| < 4, 0 \leq \arg(z) < \pi/4, \operatorname{Im}(z) < 2\}$$

razdalja od izhodišča med 1 in 4 — y-koordinate pod 2

Z območjem A naredimo naslednjo transformacijo:

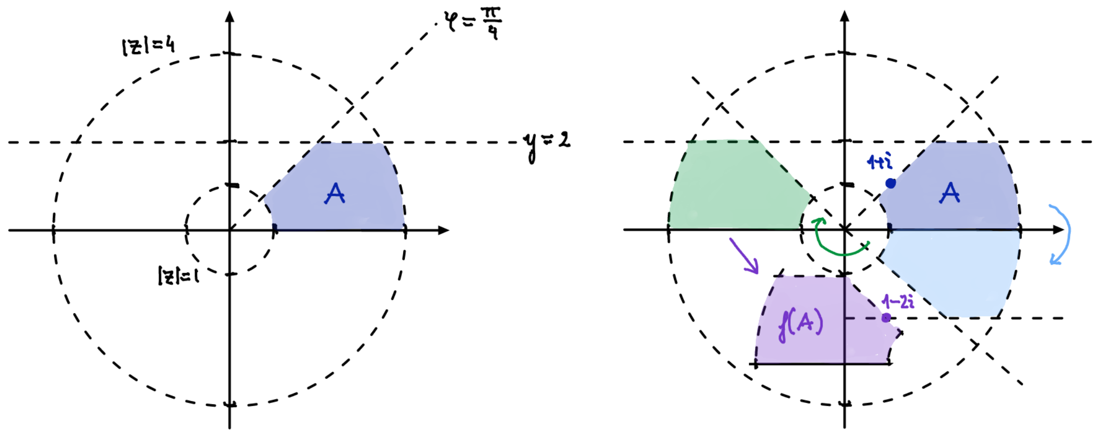
(a) prezrcalimo ga preko osi $\operatorname{Re}(z)$,

(b) zavrtimo ga okoli števila 0 za kot π,

(c) premaknemo ga za 2 v desno in 3 navzdol.

Zapiši predpis $z \mapsto f(z)$, ki opravi to kompleksno transformacijo. Nariši tudi $f(A)$ in ugotovi, kam se preslika število $1 + i$.

Rešitev: $z \mapsto -\overline{z} + 2 - 3i$, $1 + i \mapsto 1 - 2i$.



- zrcaljenje preko Re-osi: $z \longmapsto \overline{z}$
- rotacija za π okrog 0: $z \longmapsto \underbrace{e^{i\pi}}_{-1} \cdot z$

$z \longmapsto -z$

- premik 2 desno in 3 dol: $z \longmapsto z + 2 - 3i$

rotacija okrog 0 za kot φ

$z \longmapsto e^{i\varphi} \cdot z$

$z \longmapsto \overline{z} \longmapsto -\overline{z} \longmapsto -\overline{z} + 2 - 3i$

$z \longmapsto -\overline{z} + 2 - 3i$

$f(z) = -\overline{z} + 2 - 3i$

$f(1+i) = -\overline{(1+i)} + 2 - 3i = -(1-i) + 2 - 3i = -1 + i + 2 - 3i = 1 - 2i$

7. * Nariši naslednjo podmnožico v $\mathbb{C}$:

$$A = \{z \in \mathbb{C}\,;\ \underbrace{|z| \leq 1}_{\text{krog z radijem 1}}, \overbrace{\operatorname{Im}(z) < \sqrt{2}/2}^{\text{pod } y < \frac{\sqrt{2}}{2}}, \underbrace{0 < \operatorname{Re}(z) < \sqrt{2}/2}_{\text{med } x=0 \text{ in } x=\frac{\sqrt{2}}{2}}\}$$

Z območjem A naredimo naslednjo transformacijo:

(a) zavrtimo ga okoli števila 1 za kot $-\pi/2$;

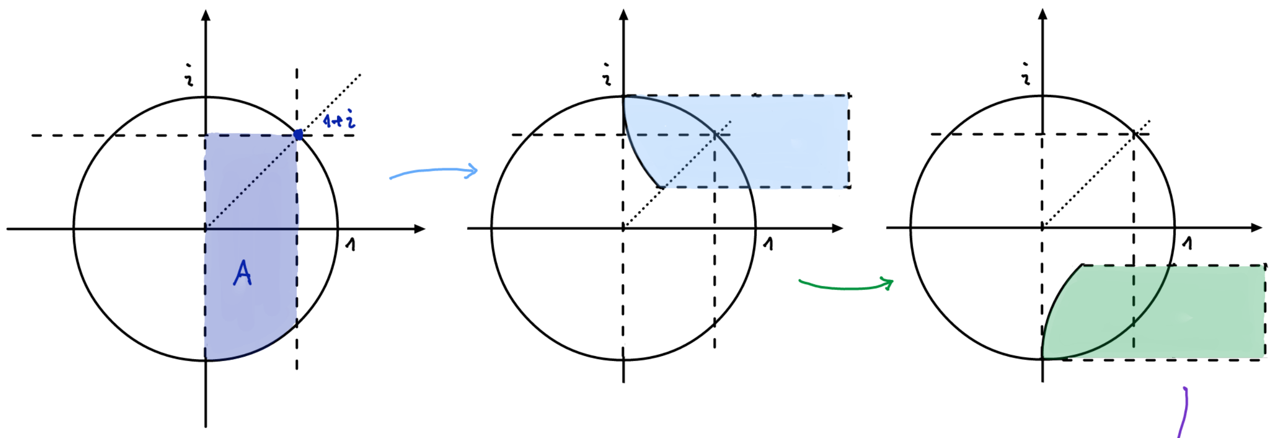
(b) prezrcalimo ga preko osi $\operatorname{Re}(z)$;

(c) premaknemo ga za 2 v levo in 1 navzdol;

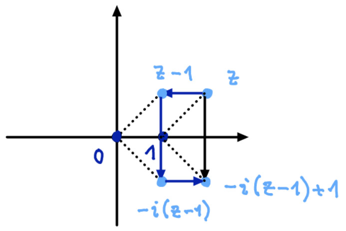
(d) zavrtimo ga okoli števila i za kot $\pi/2$.

Zapiši predpis $z \mapsto f(z)$, ki opravi to kompleksno transformacijo. Nariši tudi $f(A)$ ter ugotovi, kam se preslika število $1+i$.

Rešitev: $z \mapsto -\overline{z} + 3$, $1 + i \mapsto 2 + i$.

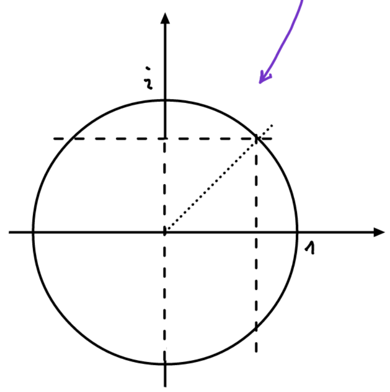


- rotacija okrog 1 za $-\frac{\pi}{2}$: $z \longmapsto -i(z-1)+1$



$z \mapsto z-1 \mapsto -i(z-1) \mapsto -i(z-1)+1$

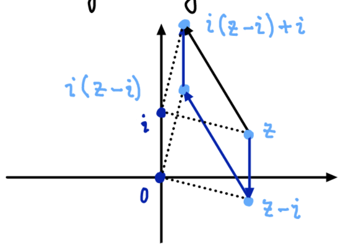
rotacija okrog w za φ =
translacija za $-w$ +
rotacija okrog 0 za φ +
translacija za w



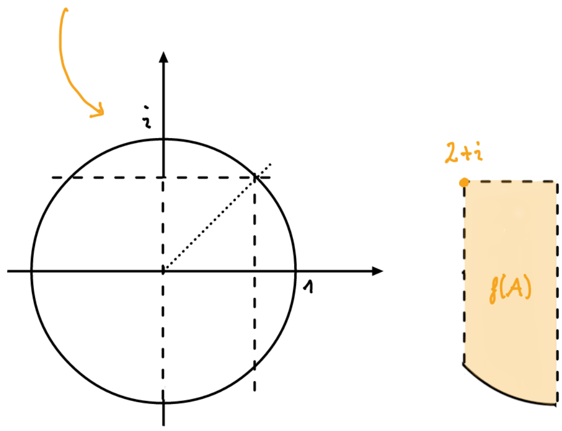
- zrcaljenje preko Re-osi : $z \longmapsto \overline{z}$

- premik 2 levo in 1 dol : $z \longmapsto z-2-i$

- rotacija okrog i za $\frac{\pi}{2}$: $z \longmapsto i(z-i)+i$



$z \mapsto z-i \mapsto i(z-i) \mapsto i(z-i)+i$



- $z \longmapsto -i(z-1)+1 \longmapsto \overline{-i(z-1)+1} = \overline{-i(z-1)}+1 = \overline{-iz+i}+1 = \overline{-iz}-i+1 \overset{*}{=} i\bar{z}-i+1 \longmapsto$

$\longmapsto i\bar{z}-i+1-2-i = i\bar{z}-2i-1 \longmapsto i\left(i\bar{z}-2i-1-i\right)+i = -\bar{z}+3-i+i = \underline{\underline{-\bar{z}+3}}$

$\underline{\underline{f(z) = -\bar{z}+3}}$

$f(1+i) = -\overline{(1+i)}+3 = -(1-i)+3 = -1+i+3 = \underline{\underline{2+i}}$

$*\quad z = x+iy$

$\left.\begin{aligned} \overline{-iz} &= \overline{-i(x+iy)} = \overline{-ix+y} = ix+y \\ i\bar{z} &= i(x-iy) = ix+y \end{aligned}\right\}$ $\boxed{\overline{-iz} = i\bar{z}}$

8. Nariši naslednje podmnožice v $\mathbb{C}$:

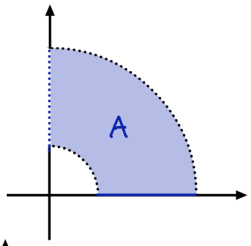
$$A = \{z \in \mathbb{C}\ ;\ 1 < |z| < 3, 0 \leq Arg(z) < \pi/2\},$$

$$B = \{z \in \mathbb{C}\ ;\ 1/3 < |z-1| < 3, 0 \leq Arg(z-1) < \pi\},$$

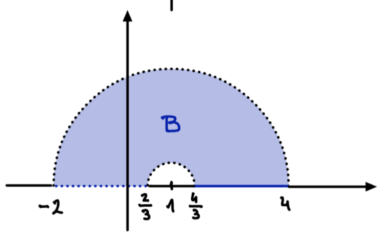
$$C = \{z \in \mathbb{C}\ ;\ 1 < |z| < 3, \pi < Arg(z) \leq 3\pi/2\}.$$

Nato poišči kompleksne transformacije, ki transformirajo A v B, A v C in C v A.

Rešitev: $f_{AB}: z \mapsto \frac{1}{3}z^2 + 1$, $f_{CA}: z \mapsto -i\bar{z}$, $f_{AC}: z \mapsto -i\bar{z}$.



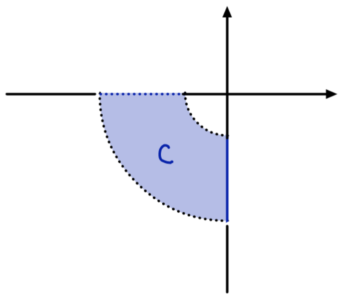
$$A = \{z \in \mathbb{C}\ ;\ \overbrace{1 < |z| < 3}^{\text{kolobar z } r=1,\, R=3}, \overbrace{0 \leq Arg(z) < \pi/2}^{\text{polarni kot v I. kvadrantu}}\},$$



$$B = \{z \in \mathbb{C}\ ;\ 1/3 < |z-1| < 3, 0 \leq Arg(z-1) < \pi\},$$

$$= \{\underbrace{z+1}_{\text{za 1 desno}} \in \mathbb{C}\ ;\ \underbrace{\tfrac{1}{3} < |z| < 3}_{\text{kolobar z } r=\frac{1}{3},\, R=3},\ \underbrace{0 \leq arg(z) < \pi}_{\text{polarni kot med 0 in } \pi}\}$$

$$C = \{z \in \mathbb{C}\ ;\ \underbrace{1 < |z| < 3}_{\text{kolobar z } r=1,\, R=3}, \underbrace{\pi < Arg(z) \leq 3\pi/2}_{\text{polarni kot v III. kvadrantu}}\}.$$



$f_{AB}: A \longrightarrow B$

$z \longmapsto z^2$ podvoji polarni kot, $[0, \frac{\pi}{2}) \rightsquigarrow [0, \pi)$
kvadrira radij, $(1,3) \rightsquigarrow (1,9)$

$z \longmapsto \frac{z}{3}$ skrčitev za faktor 3, radij $(1,9) \rightsquigarrow (\frac{1}{3}, 3)$

$z \longmapsto z^2 \longmapsto \frac{z^2}{3} \quad \rightsquigarrow \quad f_{AB}(z) = \frac{z^2}{3}$

$f_{AC}: A \longrightarrow C$

$z \longmapsto \bar{z}$ zrcaljenje čez Re-os

$z \longmapsto -iz$ rotacija okrog 0 za $-\frac{\pi}{4}$ $(-90°)$

$z \longmapsto \bar{z} \longmapsto -i\bar{z} \quad \rightsquigarrow \quad f_{AC}(z) = -i\bar{z}$

$f_{CA}: C \longrightarrow A$

zrcaljenje čez Re-os in rotacija okrog 0 za $-\frac{\pi}{4}$ $(-90°)$ $\rightsquigarrow$ $f_{CA}(z) = -i\bar{z}$