Ime in priimek

Vpisna številka

1	
2	
3	
4	
Σ	

Linearna algebra: računski izpit

5. september 2023

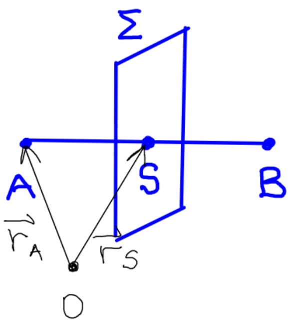
Čas pisanja: 90 minut. Dovoljena je uporaba dveh listov velikosti A4 z obrazci. Uporaba elektronskih pripomočkov ni dovoljena. Rezultati bodo objavljeni na *ucilnica.fri.uni-lj.si*. **Vse odgovore dobro utemelji!**

1. naloga (25 točk)

Dani sta točki $A(1,-1,-3)$ in $B(-1,2,2)$ ter premica $p: 1+x=y=z-4$.

$$p: \begin{aligned} x &= -1+t \\ y &= t \\ z &= 4+t \end{aligned} \quad t \in \mathbb{R}$$

a) (8) Določi enačbo ravnine Σ, ki je pravokotna na daljico $\overline{AB}$ in poteka skozi njeno razpolovišče.



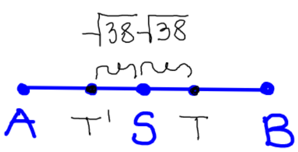
$$\vec{r}_S = \vec{r}_A + \frac{1}{2}\vec{AB} = \frac{1}{2}(\vec{r}_A + \vec{r}_B) = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{m} = \vec{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$$

$$\vec{m}\cdot\vec{r} = \vec{m}\cdot\vec{r}_S$$

$$\boxed{\Sigma: -2x+3y+5z=-1}$$

b) (9) Poišči enačbo ravnine Π, ki je vzporedna z ravnino Σ in je od nje oddaljena za $\sqrt{38}$ enot. Ali obstaja le ena taka ravnina? Utemelji.



Obstajata dve taki ravnini Π in Π', eno poteka skozi T, druga skozi T' (na sliki).

$$\vec{r}_T = \vec{r}_S + \sqrt{38}\frac{\vec{AB}}{\|\vec{AB}\|} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \frac{\sqrt{38}}{\sqrt{38}}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{7}{2} \\ \frac{9}{2} \end{bmatrix}$$

$$\vec{r}_{T'} = \vec{r}_S - \sqrt{38}\frac{\vec{AB}}{\|\vec{AB}\|} = \begin{bmatrix} 2 \\ -\frac{5}{2} \\ -\frac{11}{2} \end{bmatrix}$$

$$\vec{m}_\Pi = \vec{m}_\Sigma = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \quad \vec{m}_\Pi = \vec{m}_{\Pi'}$$

$$\Pi: \vec{m}_\Pi \cdot \vec{r} = \vec{m}_\Pi \cdot \vec{r}_T$$

$$\Pi': \vec{m}_{\Pi'} \cdot \vec{r} = \vec{m}_{\Pi'} \cdot \vec{r}_{T'}$$

$$\boxed{\Pi: -2x+3y+5z=37}$$

$$\Pi': -2x+3y+5z=-39$$

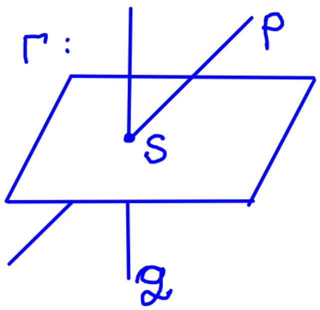
c) (8) Izračunaj presečišče S ravnine $\Gamma: 5x - 2y + 4z = -3$ in premice p ter določi enačbo premice q, ki poteka skozi S je pravokotna na ravnino Γ.

Presečišče: $5(-1+t) - 2t + 4(4+t) = -3$

$$-5 + 5t - 2t + 16 + 4t = -3$$

$$7t = -14$$

$$t = -2 \rightarrow \boxed{S(-3,-2,2)}$$



$$\vec{s}_q = \vec{n}_\Gamma = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

$$\boxed{q: \begin{array}{l} x = -3 + 5t \\ y = -2 - 2t \\ z = 2 + 4t \end{array}, \ t \in \mathbb{R}}$$

2. naloga (25 točk)

Naj bo

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 7 & -1 \\ 2 & 1 & 3 \\ 5 & 7 & 3 \end{bmatrix}, \quad \vec{b}_1 = \begin{bmatrix} 0 \\ \alpha - 3 \\ \alpha - 3 \\ \alpha - 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} \beta - 1 \\ \beta \\ -4 \\ -1 \end{bmatrix},$$

kjer sta $\alpha, \beta \in \mathbb{R}$.

a) (10) Poišči bazo stolpčnega prostora matrike A. Kolikšna je dimenzija ničelnega prostora matrike A?

$$A^T = \begin{bmatrix} 2 & 3 & 2 & 5 \\ 5 & 7 & 1 & 7 \\ -1 & -1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & 1 & 8 & 11 \\ 0 & 1 & 8 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 2 & 5 \\ 0 & 1 & 8 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B}_{C(A)} = \left\{ \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \\ 11 \end{bmatrix} \right\} \qquad \dim C(A) = 2$$

$$\underset{\substack{\parallel \\ 2}}{\dim C(A)} + \dim N(A) = \underset{\substack{\parallel \\ 3}}{\text{št. stolpcev } A}$$

$$\Downarrow$$

$$\dim N(A) = 1$$

b) (8) Za katere $\alpha \in \mathbb{R}$ je sistem $A\vec{x} = \vec{b}_1$ rešljiv? Za dobljen $\alpha \in \mathbb{R}$ zapiši vse rešitve tega sistema.

$$\left[\begin{array}{ccc|c} 2 & 5 & -1 & 0 \\ 3 & 7 & -1 & \alpha-3 \\ 2 & 1 & 3 & \alpha-3 \\ 5 & 7 & 3 & \alpha-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & -1 & 0 \\ 0 & 1 & -1 & -2\alpha+6 \\ 0 & 4 & -4 & -\alpha+3 \\ 0 & 11 & -11 & -2\alpha+6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 5 & -1 & 0 \\ 0 & 1 & -1 & -2\alpha+6 \\ 0 & 0 & 0 & 7\alpha-21 \\ 0 & 0 & 0 & 20\alpha-60 \end{array}\right]$$

Sistem je rešljiv, če je $7\alpha - 21 = 0$ in $20\alpha - 60 = 0$.

$$\Updownarrow$$

$$\underline{\underline{\alpha = 3}}$$

$\underline{\alpha = 3:}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 5 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_2 = x_3 \\ \to 2x_1 + 5x_2 - x_3 = 0 \\ \quad 2x_1 = -4x_2 \\ \quad\quad x_1 = -2x_2 \end{array}$$

$$\text{Rešitve: } \left\{ \begin{bmatrix} -2x_2 \\ x_2 \\ x_2 \end{bmatrix} ; x_2 \in \mathbb{R} \right\}$$

c) (7) Poišči tak $\beta \in \mathbb{R}$, da bo $\vec{b_2}$ element stolpčnega prostora matrike A.

$$\left[\begin{array}{cc|c} 2 & 0 & \beta-1 \\ 3 & 1 & \beta \\ 2 & 8 & -4 \\ 5 & 11 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & -2 \\ 5 & 11 & -4 \\ 2 & 0 & \beta-1 \\ 3 & 1 & \beta \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & -8 & \beta+3 \\ 0 & -11 & \beta+6 \end{array}\right] \sim$$

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & \beta-5 \\ 0 & 0 & \beta-5 \end{array}\right] \end{array} \rightsquigarrow \underline{\underline{\beta = 5}}$$

$x_2 = -1$

$x_1 = -2 - 4x_2 = 2$

$$\vec{b_2} = 2 \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 8 \\ 11 \end{bmatrix}$$

3. naloga (25 točk)

Preslikavi $\phi, \psi \colon \mathbb{R}^3 \to \mathbb{R}^3$ sta dani s predpisoma

$$\phi([x,y,z]^{\mathsf{T}}) = [|x|, 0, |y|]^{\mathsf{T}} \text{ ter } \psi(\vec{x}) = (\vec{x} \cdot \vec{v})\,\vec{v}.$$

a) (15) Pokaži, da je preslikava ψ linearna, preslikava ϕ pa ne.

ψ linearna:

- $\psi(\vec{x}+\vec{y}) = ((\vec{x}+\vec{y})\cdot\vec{v})\vec{v} = (\vec{x}\cdot\vec{v} + \vec{y}\cdot\vec{v})\vec{v} =$
  $= (\vec{x}\cdot\vec{v})\vec{v} + (\vec{y}\cdot\vec{v})\vec{v} = \psi(\vec{x}) + \psi(\vec{y})$ ✓
- $\psi(\alpha\vec{x}) = (\alpha\vec{x}\cdot\vec{v})\vec{v} = \alpha(\vec{x}\cdot\vec{v})\vec{v} = \alpha\cdot\psi(\vec{x})$ ✓

ϕ ni linearno, saj ni aditivna:

$\vec{x} = (x_1, x_2, x_3)^T$, $\vec{y} = (y_1, y_2, y_3)^T$

$$\phi(\vec{x}+\vec{y}) = (|x_1+y_1|, 0, |x_2+y_2|) \neq$$
$$(|x_1|, 0, |x_2|) + (|y_1|, 0, |y_2|) = \phi(\vec{x}) + \phi(\vec{y})$$

Protiprimer: $\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\phi(\vec{x}+\vec{y}) = (0,0,0)$$
$$\phi(\vec{x}) + \phi(\vec{y}) = (1,0,1) + (1,0,1) = (2,0,2)$$

b) (10) Za $\vec{v} = [1,2,3]^T$ poišči matriko, ki pripada preslikavi ψ glede na standardno bazo prostora $\mathbb{R}^3$.

$$\psi(\vec{e}_1) = (\vec{e}_1 \cdot \vec{v})\vec{v} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2 + 3\vec{e}_3$$

$$\psi(\vec{e}_2) = (\vec{e}_2 \cdot \vec{v})\vec{v} = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} =$$
$$= 2 \cdot \vec{e}_1 + 4\vec{e}_2 + 6\vec{e}_3$$

$$\psi(\vec{e}_3) = (\vec{e}_3 \cdot \vec{v})\vec{v} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} =$$
$$= 3\vec{e}_1 + 6\vec{e}_2 + 9\vec{e}_3$$

$$A_\psi = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

4. naloga (25 točk)

Naj bo

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}.$$

a) (15) Poišči vse lastne vrednosti in pripadajoče lastne vektorje matrike A.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & 5-\lambda & -4 \\ -2 & -4 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & -2 \\ 0 & 1-\lambda & 1-\lambda \\ -2 & -4 & 5-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 2 & -2 \\ 0 & 1 & 1 \\ -2 & -4 & 5-\lambda \end{vmatrix} = (1-\lambda)\big((2-\lambda)(5-\lambda+4) - 2\cdot 4\big)$$

$$= (1-\lambda)(\lambda^2-11\lambda+10)$$

$$= (1-\lambda)(\lambda-10)(\lambda-1)$$

$$\underline{\underline{\lambda_{1,2}=1}}, \quad \underline{\underline{\lambda_3=10}}$$

$\underline{\lambda_{1,2}=1:}$

$$A-I=\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1=-2x_2+2x_3 \quad \begin{bmatrix} -2x_2+2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underline{\underline{\vec{v}_1=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}}$$

$\underline{\lambda_3=10:}$

$$A-10I=\begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} -4 & 1 & -1 \\ 0 & -9 & -9 \\ 0 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} -4 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2=-x_3$$
$$-4x_1=-x_2+x_3$$
$$-4x_1=2x_3$$
$$x_1=-\tfrac{1}{2}x_3$$

$$\begin{bmatrix} -\frac{1}{2}x_3 \\ -x_3 \\ x_3 \end{bmatrix} \qquad \underline{\underline{\vec{v}_3=\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}}}$$

b) **(10)** Izberi ortogonalna lastna vektorja matrike A ter ju označi z $\vec{u}$ in $\vec{v}$. Dopolni množico $\{\vec{u},\vec{v}\}$ do ortogonalne baze prostora $\mathbb{R}^3$.

npr.

$$\underset{\vec{u}}{\underset{\parallel}{\vec{v}_2}} \perp \underset{\vec{v}}{\underset{\parallel}{\vec{v}_3}} \qquad \vec{v}_4=\vec{u}\times\vec{v}=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\times\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}=\begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}$$

$\vec{v}_2, \vec{v}_3, \vec{v}_4$ so paroma pravokotni, sedaj jih še normiramo:

$$\vec{q}_1=\frac{1}{\sqrt{45}}\begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}=\frac{1}{3\sqrt{5}}\begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}, \quad \vec{q}_2=\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{q}_3=\frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ je ONB za $\mathbb{R}^3$