Mathematical Modelling Exam

June 29th, 2023

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let $A \in \mathbb{R}^{n\times m}$ be a given matrix and $X \in \mathbb{R}^{m\times n}$ a matrix satisfying the following property:

$$\text{If the sytem } Ax = b \text{ is solvable, then } x = Xb \text{ is one of the solutions.} \tag{1}$$

   (a) Show that X is a generalized inverse of A.

   (b) Give an example of a matrix $A \in \mathbb{R}^{2\times 2}$ and a matrix X such that (1) is satisfied, but X is not the Moore-Penrose inverse of A.

Solution.

(a) We have to check that $AXA = A$. Equivalently, for every $x \in \mathbb{R}^m$ it should hold that $AXAx = Ax$. Let $x \in \mathbb{R}^m$ be arbitrary. Denote $b := Ax$. Hence,

$$AXAx = AX(Ax) = AXb = A(Xb) = Ax = b,$$

where we used the assumption (1) in the fourth equality. So $AXAx = Ax$ and X is a generalized inverse of A.

(b) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it holds that $AXA = A$, but $X \neq A^\dagger = A$.

2. Let $f(x, y, z) = \left(2 - \sqrt{x^2 + y^2}\right)^2 + z^2 - 1$ be a function of three variables.

   (a) Check that every point on the torus with the center of the hole being the origin, the distance from the origin to the center of the torus tube being 2 and the radius of the tube being 1, is a solution of the equation $f(x, y, z) = 0$.

   (b) Perform one step of Gauss-Newton method to approximate the point on the torus $f(x, y, z) = 0$ given the initial approximation $(x_0, y_0, z_0) = \big(1, 1, 1\big)$.

Solution.

(a) A parametrization of the torus from the exercise is

$$r(u, v) = \big((2 + \cos v) \cos u, (2 + \cos v) \sin u, \sin v\big),$$

where $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $v \in [0, 2\pi]$. We plug this into $f(x, y, z)$ and check that $f(x, y, z) = 0$:

$$\begin{aligned}
& f((2 + \cos v) \cos u, (2 + \cos v) \sin u, \sin v) \\
&= \left(2 - \sqrt{(2 + \cos v)^2 \cos^2 u + (2 + \cos v)^2 \sin^2 u}\right)^2 + \sin^2 v - 1 \\
&= \left(2 - \sqrt{(2 + \cos v)^2 (\cos^2 u + \sin^2 u)}\right)^2 + \sin^2 v - 1 \\
&= \left(2 - \sqrt{(2 + \cos v)^2}\right)^2 + \sin^2 v - 1 \\
&= (2 - 2 - \cos v)^2 + \sin^2 v - 1 \\
&= \cos^2 v + \sin^2 v - 1 \\
&= 0.
\end{aligned}$$

(b) The iteration procedure for the Gauss-Newton method for the function $f : \mathbb{R}^3 \to \mathbb{R}$ is

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - \big((Jf)(x_0, y_0, z_0)\big)^\dagger f(x_0, y_0, z_0).$$

We have that

$$(Jf)(x, y, z) = \begin{bmatrix} f_x(x, y, z) & f_y(x, y, z) & f_z(x, y, z) \end{bmatrix},$$

where

$$\begin{aligned} f_x(x, y, z) &= \begin{bmatrix} 2 \cdot (2 - \sqrt{x^2 + y^2}) \cdot (-\frac{1}{2}) \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot 2x \end{bmatrix}, \\ f_y(x, y, z) &= \begin{bmatrix} 2 \cdot (2 - \sqrt{x^2 + y^2}) \cdot (-\frac{1}{2}) \cdot \frac{1}{\sqrt{x^2+y^2}} \cdot 2y \end{bmatrix}, \\ f_z(x, y, z) &= \begin{bmatrix} 2z \end{bmatrix}, \end{aligned}$$

Hence, $(Jf)(1, 1, 1) = \big[-(2 - \sqrt{2})\sqrt{2}, -(2 - \sqrt{2})\sqrt{2}, 2\big]$ and denoting $p = (1, 1, 1)$,

$$\begin{aligned} (Jf)^\dagger(p) &= \big((Jf)(p)((Jf)(p))^T\big)^{-1}(Jf)^T(p) \\ &= \frac{1}{28 - 16\sqrt{2}} \begin{bmatrix} -(2 - \sqrt{2})\sqrt{2} \\ -(2 - \sqrt{2})\sqrt{2} \\ 2 \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{28 - 16\sqrt{2}} \begin{bmatrix} -(2 - \sqrt{2})\sqrt{2} \\ -(2 - \sqrt{2})\sqrt{2} \\ 2 \end{bmatrix} (2 - \sqrt{2})^2 \\ &\approx \begin{bmatrix} 1.05 \\ 1.05 \\ 0.87 \end{bmatrix}. \end{aligned}$$

3. Let

$$f(u, v) = (u, v, u^2 - v^2), \quad (u, v) \in \mathbb{R}^2$$

be the parametric surface Π.

(a) Sketch few coordinate curves for each parameter u and v.

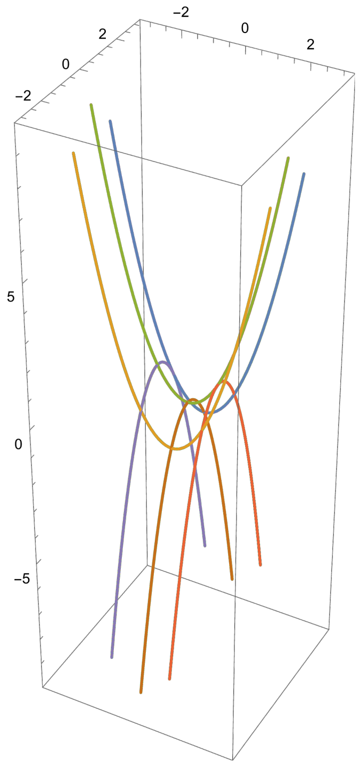
(b) Sketch the surface Π.

(c) Determine the parametric equation of the tangent plane at every point of Π.

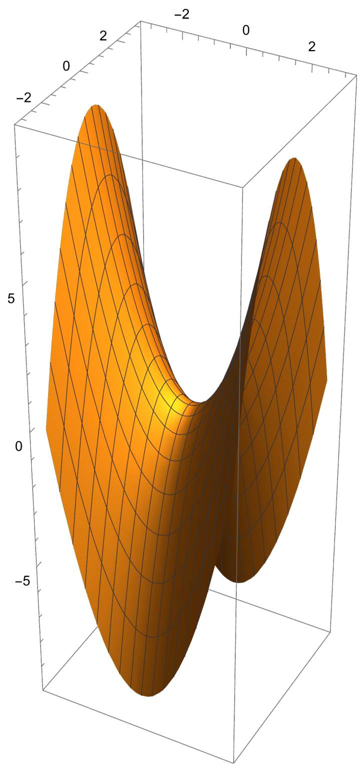
(d) Write the expression for the length of the curve $\gamma(t) = (t, t^2, t^2 - t^4)$ on Π between the points $(0, 0, 0)$ and $(1, 1, 0)$ and approximate it with a simple trapezoid rule.

Solution.

(a) Some coordinate curves are:



(b) The sketch of the surface is:



(c) The tangent plane to Π in the point $f(u_0, v_0)$ is

$$
\begin{aligned}
L(\lambda, \mu) &= f(u_0, v_0) + \lambda \cdot f_u(u_0, v_0) + \mu \cdot f_v(u_0, v_0) \\
&= \begin{bmatrix} u_0 \\ v_0 \\ u_0^2 - v_0^2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 2u_0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ -2v_0 \end{bmatrix} \\
&= \begin{bmatrix} u_0 + \lambda \\ v_0 + \mu \\ u_0^2 - v_0^2 + \lambda 2u_0 - \mu 2v_0 \end{bmatrix}.
\end{aligned}
$$

(d) The expression for the length of the curve $\gamma(t)$ between the point corresponding to $t = 0$ and $t = 1$ is

$$\ell = \int_0^1 \|\gamma'(t)\|\, dt = \int_0^1 \left\|(1, 2t, 2t - 4t^3)\right\| dt = \int_0^1 \sqrt{1 + 4t^2 + (2t - 4t^3)^2} dt.$$

Using a simple trapeziod rule an approximation of ℓ is

$$\ell \approx \frac{1}{2}\left(\|\gamma'(0)\| + \|\gamma'(1)\|\right) = \frac{1}{2}\left(\sqrt{1} + \sqrt{9}\right) = 2.$$

4. Let

$$y' + 2y = 2 - e^{-4x}$$

be the differential equation (DE) with the initial condition $y(0) = 1$.

(a) Solve the DE exactly.

(b) Use Euler's Method with a step size $h = 0.1$ to find approximate values of the solution at $x = 0.1$ and $x = 0.2$. Compare these approximations with the exact values of the solution at these points.

Solution.

(a) For the homogeneous we use separation of variables to obtain

$$\frac{dy}{y} = -2dx$$

and hence $\log|y| = -2x + C$, $C \in \mathbb{R}$. Further on, $y_h(x) = Ke^{-2x}$, $K \in \mathbb{R}$. For the particular part we use

$$y_p(x) = A + Be^{-4x}$$

and obtain

$$y_p' + 2y = -4Be^{-4x} + 2(A + Be^{-4x}) = 2 - e^{-4x}.$$

Comparing both sides of the equation we get $A = 1$ and $B = \frac{1}{2}$. Hence,

$$y(x) = y_h(x) + y_p(x) = Ke^{-2x} + 1 + \frac{1}{2}e^{-4x}.$$

Using the initial condition $y(0) = 1$ it follows that $K = -\frac{1}{2}$.

(b) Using Euler's method to approximate $y(0.1), y(0.2)$ we get

$$\begin{aligned}
y(0.1) &= y(0) + 0.1 \cdot y'(0) = y(0) + 0.1(-2y(0) + 2 - e^{-0}) = 1 - 0.1 = 0.9,\\
y(0.2) &= y(0.1) + 0.1 \cdot y'(0.1) = y(0.1) + 0.1(-2y(0.1) + 2 - e^{-0.4})\\
&= 0.9 + 0.1(-1.8 + 2 - e^{-0.4}) \approx 0.85.
\end{aligned}$$

Exact solutions are

$$\begin{aligned}
y(0.1) &= -\frac{1}{2}e^{-0.2} + 1 + \frac{1}{2}e^{-0.4} = 0.93,\\
y(0.2) &= -\frac{1}{2}e^{-0.4} + 1 + \frac{1}{2}e^{-0.8} = 0.89.
\end{aligned}$$