Mathematical Modelling Exam

June 7th, 2023

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 90 minutes to solve the problems.

1. **[15 points]** Let

$$A = \begin{bmatrix} 5 & -1 \\ 1 & -5 \\ 5 & -1 \\ 1 & -5 \end{bmatrix} \qquad \text{and} \qquad c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

be a matrix and a vector. The singular value decomposition (SVD) of A is equal to

$$U\Sigma V^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T.$$

(a) Compute the lengths α_1, α_2 of both projections of the vector c to the left singular vectors of A and compute the projection p of c to the orthogonal complement of the span of the left singular vectors of A.

(b) In the notation above show that for any matrix $A \in \mathbb{R}^{4\times 2}$ it holds that

$$\begin{bmatrix} A & c \end{bmatrix} = \begin{bmatrix} U & \frac{p}{\|p\|} \end{bmatrix} \left[\begin{array}{c|c} \Sigma & \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \\ \hline 0_{1\times 2} & \|p\| \end{array}\right] \left[\begin{array}{c|c} V & 0_{2\times 1} \\ \hline 0_{1\times 2} & 1 \end{array}\right]^T. \tag{2}$$

Here, $\|p\|$ stands for the usual Euclidean norm of the vector p and $0_{i\times j}$ stands for the $i \times j$ matrix with zero entries.

(c) Using (1b) for the given matrix A and the vector c from (1), compute the singular values of the matrix $\begin{bmatrix} A & c \end{bmatrix}$.

Hint: Observe that $\begin{bmatrix} U & \frac{p}{\|p\|} \end{bmatrix}$ and $\left[\begin{array}{c|c} V & 0_{2\times 1} \\ \hline 0_{1\times 2} & 1 \end{array}\right]$ are orthogonal matrices and hence you only need to compute the singular values of the middle matrix in the factorization (2). For all points give a short explanation why this last conclusion holds.

Solution.

(a) The singular vectors of A are

$$u_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \qquad \text{and} \qquad u_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T.$$

The projections of c to u_1, u_2 have lengths

$$\alpha_1 = \langle u_1, c\rangle = \frac{1}{2} - \frac{1}{2} = 0, \qquad \alpha_2 = \langle u_2, c\rangle = \frac{1}{2} + \frac{1}{2} = 1.$$

The projection p of c to the orthogonal complement of the span of u_1 and u_2 is equal to

$$p = c - \alpha_1 u_1 - \alpha_2 u_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^T.$$

(b) We have that

$$
\begin{aligned}
&\begin{bmatrix} U & \frac{p}{\|p\|} \end{bmatrix} \left[\begin{array}{c|c} \Sigma & \begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} \\ \hline 0_{2\times 1} & \|p\| \end{array}\right] \left[\begin{array}{c|c} V & 0_{2\times 1} \\ \hline 0_{1\times 2} & 1 \end{array}\right]^T \\
&= \begin{bmatrix} U\Sigma & U\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} + p \end{bmatrix} \left[\begin{array}{c|c} V & 0_{2\times 1} \\ \hline 0_{1\times 2} & 1 \end{array}\right]^T \\
&= \begin{bmatrix} U\Sigma & c \end{bmatrix} \left[\begin{array}{c|c} V^T & 0_{2\times 1} \\ \hline 0_{1\times 2} & 1 \end{array}\right] \\
&= \begin{bmatrix} U\Sigma V^T & c \end{bmatrix} \\
&= \begin{bmatrix} A & c \end{bmatrix},
\end{aligned}
$$

where we used that

$$
U\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} + p = \alpha_1 u_1 + \alpha_2 u_2 + p = c
$$

in the third equality.

(c) Since the SVD is unique (up to permutation of singular values and vectors), denoting the SVD of the matrix $\left[\begin{array}{c|c} \Sigma & \begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} \\ \hline 0_{1\times 2} & \|p\| \end{array}\right]$ by $U'\Sigma(V')^T$, it follows that

$$
\left(\begin{bmatrix} U & \frac{p}{\|p\|} \end{bmatrix} U'\right)\Sigma\left(\left[\begin{array}{c|c} V & 0_{2\times 1} \\ \hline 0_{1\times 2} & 1 \end{array}\right] V'\right)^T
$$

is the SVD of $\begin{bmatrix} A & c \end{bmatrix}$. We have that

$$
\left[\begin{array}{c|c} \Sigma & \begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} \\ \hline 0_{1\times 2} & \|p\| \end{array}\right] = \begin{bmatrix} 6\sqrt{2} & 0 & 0 \\ 0 & 4\sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} =: M
$$

and hence the singular values are square roots of the zeros of

$$
\begin{aligned}
\det(M^T M - \lambda I_3) &= \det\left(\begin{bmatrix} 72-\lambda & 0 & 0 \\ 0 & 32-\lambda & 4\sqrt{2} \\ 0 & 4\sqrt{2} & 2-\lambda \end{bmatrix}\right) \\
&= (72-\lambda)(\lambda^2 - 34\lambda + 32).
\end{aligned}
$$

Hence, $\sigma_1 = \sqrt{\lambda_1} = \sqrt{72}$, $\sigma_2 = \sqrt{\lambda_2} = 17 + \sqrt{257}$, $\sigma_1 = \sqrt{\lambda_3} = 17 - \sqrt{257}$.

2. **[10 points]** Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ be a diagonal matrix with $0 < d_1 < \ldots < d_n$ and let $z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^T \in \mathbb{R}^n$ be a vector. We form a matrix $M = \begin{bmatrix} D & z \end{bmatrix}$, i.e.,

$$
M = \begin{bmatrix} d_1 & & & & z_1 \\ & d_2 & & & z_2 \\ & & \ddots & & \vdots \\ & & & d_n & z_n \end{bmatrix}
$$

It turns out that nonzero singular values $\sigma_1, \ldots, \sigma_n$ of the matrix M are solutions of the nonlinear equation

$$
1 + \sum_{i=1}^{n} \frac{z_i^2}{d_i^2 - w^2} = 0
$$

and they satisfy the interlacing property

$$0 < d_1 < w_1 < d_2 < w_2 < \ldots < d_n < w_n < d_n + \|z\|.$$

Let $n = 3$, $d_1 = 1, d_2 = 2, d_3 = 3$ and $z_1 = z_2 = z_3 = 1$.

Task: Perform one step of Newton's method with a suitably chosen initial approximation $w^{(0)}$ to estimate the smallest and the largest singular values σ_1, σ_3, i.e., repeat the procedure twice with different, meaningfully chosen initial approximations.

Solution. To apply the Newton's method we need to compute the derivative of the function

$$f(w) := 1 + \frac{1}{1-w^2} + \frac{1}{4-w^2} + \frac{1}{9-w^2},$$

i.e.,

$$f'(w) = \frac{2w}{(1-w^2)^2} + \frac{2w}{(4-w^2)^2} + \frac{2w}{(9-w^2)^2}.$$

Then one step of Newton's method is

$$w^{(n+1)} = w^{(n)} - \frac{f(w^{(n)})}{f'(w^{(n)})}.$$

By the interlacing property

$$1 < w_1 < 2 < w_2 < 3 < w_3 < 3 + \sqrt{3}.$$

To estimate w_1 it makes sense to use $w_1^{(0)} = \frac{3}{2}$. Since $f(1.5) = 0.920$ and $f'(1.5) = 2.965$, it follows that $w_1^{(1)} = 1.5 - \frac{0.920}{2.965} = 1.190$.

To estimate w_3 it makes sense to use $w_3^{(0)} = 3.5$. Since $f(3.5) = 0.482$ and $f'(3.5) = 0.821$, it follows that $w_3^{(1)} = 3.5 - \frac{0.482}{0.821} = 2.913$.

3. **[15 points]** Let $f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$, $u_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], u_2 \in [0, 2\pi)$ be a parametrization of the torus:

$$f(u_1,u_2) = \big(\big(2+\cos(u_1)\big)\cos(u_2), \big(2+\cos(u_1)\big)\sin(u_2), \sin(u_1)\big).$$

(a) Compute the matrix G, called the metric tensor, and its inverse G^{-1}:

$$G = \begin{bmatrix} \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_1} & \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_2} \\ \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_2}\frac{\partial f_k}{\partial u_1} & \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_2}\frac{\partial f_k}{\partial u_2} \end{bmatrix}, \qquad G^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}.$$

(b) For $i,j,k = 1,2$, the so-called Christoffel symbols Γ_{ij}^k are defined by:

$$\Gamma_{ij}^k = \sum_{\ell=1}^{2} \left\langle \begin{bmatrix} \frac{\partial^2 f_1}{\partial u_i \partial u_j} \\ \frac{\partial^2 f_2}{\partial u_i \partial u_j} \\ \frac{\partial^2 f_3}{\partial u_i \partial u_j} \end{bmatrix}, \begin{bmatrix} \frac{\partial f_1}{\partial u_\ell} \\ \frac{\partial f_2}{\partial u_\ell} \\ \frac{\partial f_3}{\partial u_\ell} \end{bmatrix} \right\rangle h_{\ell k},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product of vectors. Compute Γ_{11}^1.

(c) The shortest paths on torus are obtained by solving the following second order system of differential equations:

$$\begin{aligned} \frac{d^2u_1}{dt^2} + \sum_{i,j=1}^{2} \Gamma^1_{ij} \frac{du_i}{dt}\frac{du_j}{dt} &= 0, \\ \frac{d^2u_2}{dt^2} + \sum_{i,j=1}^{2} \Gamma^2_{ij} \frac{du_i}{dt}\frac{du_j}{dt} &= 0. \end{aligned} \tag{3}$$

Except Γ^1_{11} you computed in (3b), the remaining Christoffel symbols are

$$\begin{aligned} &\Gamma^2_{11} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^2_{12} = \Gamma^2_{21} = -\frac{\sin(u_1)}{2+\cos(u_1)}, \\ &\Gamma^1_{22} = (2+\cos(u_1))\sin(u_2), \quad \Gamma^2_{22} = 0. \end{aligned}$$

Write down the system (3) explicitly.

(d) Translate the system from (3c) to the first order system of differential equations by introducing new variables

$$x_1(t) = u_1(t),\ x_2(t) = \frac{du_1}{dt},\ x_3(t) = u_2(t),\ x_4(t) = \frac{du_2}{dt}.$$

(e) For the initial point $(u_1(0), u_2(0)) = (0,0)$ and derivatives $\left(\frac{du_1}{dt}(0), \frac{du_2}{dt}(0)\right) = (1,1)$ perform one step of Euler's method on the system from (3d) with a step size $h = 0.1$.

Solution. We have that

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \\ \frac{\partial f_3}{\partial u_1} \end{bmatrix} = \begin{bmatrix} -\sin(u_1)\cos(u_2) \\ -\sin(u_1)\sin(u_2) \\ \cos(u_1) \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_2} \end{bmatrix} = \begin{bmatrix} -(2+\cos(u_1))\sin(u_2) \\ (2+\cos(u_1)\cos(u_2) \\ 0 \end{bmatrix}.$$

(a) The following calculations hold:

$$\begin{aligned} \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_1} &= \sin^2(u_1)\cos^2(u_2) + \sin^2(u_1)\sin^2(u_2) + \cos^2(u_1) \\ &= \sin^2(u_1)\big(\cos^2(u_2) + \sin^2(u_2)\big) + \cos^2(u_1) = \sin^2(u_1) + \cos^2(u_1) = 1, \\ \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_2} &= \sin(u_1)\cos(u_2)(2+\cos(u_1))\sin(u_2) \\ &\qquad - \sin(u_1)\sin(u_2)(2+\cos(u_1))\cos(u_2) = 0, \\ \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_2}\frac{\partial f_k}{\partial u_2} &= (2+\cos(u_1))^2\sin(u_2)^2 + (2+\cos(u_1))^2\cos(u_2)^2 \\ &= (2+\cos(u_1))^2(\sin^2(u_2) + \cos^2(u_2)) = (2+\cos(u_1))^2. \end{aligned}$$

Hence,

$$G = \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(u_1))^2 \end{bmatrix} \quad \text{and} \quad G^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(u_1))^{-2} \end{bmatrix}.$$

(b) We have that

$$\begin{bmatrix} \dfrac{\partial^2 f_1}{\partial u_1 \partial u_1} \\ \dfrac{\partial^2 f_2}{\partial u_1 \partial u_1} \\ \dfrac{\partial^2 f_3}{\partial u_1 \partial u_1} \end{bmatrix} = \begin{bmatrix} -\cos(u_1)\cos(u_2) \\ -\cos(u_1)\sin(u_2) \\ -\sin(u_1) \end{bmatrix}.$$

Hence,

$$\begin{aligned} \Gamma_{11}^1 &= \sum_{\ell=1}^{2} \left\langle \begin{bmatrix} \dfrac{\partial^2 f_1}{\partial u_1 \partial u_1} \\ \dfrac{\partial^2 f_2}{\partial u_1 \partial u_1} \\ \dfrac{\partial^2 f_3}{\partial u_1 \partial u_1} \end{bmatrix}, \begin{bmatrix} \dfrac{\partial f_1}{\partial u_\ell} \\ \dfrac{\partial f_2}{\partial u_\ell} \\ \dfrac{\partial f_3}{\partial u_\ell} \end{bmatrix} \right\rangle h_{\ell 1} = \left\langle \begin{bmatrix} \dfrac{\partial^2 f_1}{\partial u_1 \partial u_1} \\ \dfrac{\partial^2 f_2}{\partial u_1 \partial u_1} \\ \dfrac{\partial^2 f_3}{\partial u_1 \partial u_1} \end{bmatrix}, \begin{bmatrix} \dfrac{\partial f_1}{\partial u_1} \\ \dfrac{\partial f_2}{\partial u_1} \\ \dfrac{\partial f_3}{\partial u_1} \end{bmatrix} \right\rangle h_{11} \\ &= \cos(u_1)\sin(u_1)\cos^2(u_2) + \cos(u_1)\sin(u_1)\sin^2(u_2) - \sin(u_1)\cos(u_1) \\ &= \cos(u_1)\sin(u_1)\big(\cos^2(u_2) + \sin^2(u_2)\big) - \sin(u_1)\cos(u_1) \\ &= \cos(u_1)\sin(u_1) - \cos(u_1)\sin(u_1) = 0. \end{aligned}$$

(c) The system is

$$\begin{aligned} \frac{d^2 u_1}{dt^2} + (2 + \cos(u_1))\sin(u_2)\frac{du_2}{dt}\frac{du_2}{dt} &= 0, \\ \frac{d^2 u_2}{dt^2} - 2\frac{\sin(u_1)}{2 + \cos(u_1)}\frac{du_1}{dt}\frac{du_2}{dt} &= 0. \end{aligned} \tag{4}$$

(d) The corresponding first order system to the system (4) is

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= -(2 + \cos(x_1))\sin(x_3)x_4^2, \\ \frac{dx_3}{dt} &= x_4, \\ \frac{dx_4}{dt} &= \frac{2\sin(x_1)}{2 + \cos(x_1)}x_2 x_4. \end{aligned} \tag{5}$$

(e) Using Euler's method on (5) with $t_0 = 0$ and $h = 0.1$ we get

$$\begin{aligned} x_1(0.1) &= x_1(0) + 0.1 \cdot x_2(0) \\ &= 0 + 0.1 \cdot 1 = 0.1, \\ x_2(0.1) &= x_2(0) - 0.1 \cdot (2 + \cos(x_1(0)))\sin(x_3(0))x_4^2(0) \\ &= 1 - 0.1 \cdot (2 + \cos(0))\sin(0)1^2 = 1, \\ x_3(0.1) &= x_3(0) + 0.1 \cdot x_4(0) \\ &= 0 + 0.1 \cdot 1 = 0.1, \\ x_4(0.1) &= x_4(0) + 0.1 \cdot \frac{2\sin(x_1(0))}{2 + \cos(x_1(0))}x_2(0)x_4(0) \\ &= 1 + 0.1 \cdot \frac{2\sin(0)}{2 + \cos(0)} = 1. \end{aligned}$$