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Obnova matematičnega znanja

1.1) Differencialne enačbe

a) Prvega reda

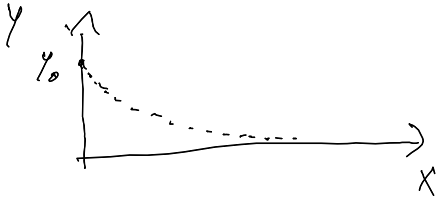
$y'(x) = -\alpha y(x)$ z.p. $y(x=0) = y_0$

nastavek $y(x) = A e^{-\lambda x}$

$-\lambda A e^{-\lambda x} = -\alpha A e^{-\lambda x} \Rightarrow \lambda = \alpha$

$y(0) = A$

Rešitev $\boxed{y(x) = y_0 e^{-\alpha x}}$



b) Drugega reda

$y'' + \alpha y' + \omega^2 y = 0$

$y(x=0) = y_0$
$y'(x=0) = v_0$

→ Harmonski oscilator $\alpha = 0$

$y'' + \omega^2 y = 0$ Nastavek $y(x) = A\sin(\alpha x) + B\cos(\alpha x)$

$y'' = -\alpha^2 [A\sin(\alpha x) + B\cos(\alpha x)] = -\omega^2 y$

$-\alpha^2 y + \omega^2 y \quad 0 \Rightarrow \alpha = \omega$

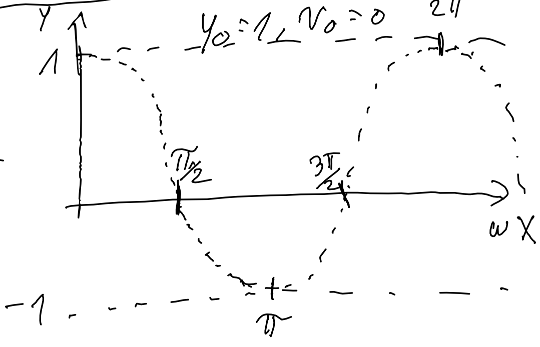
$$Y_0 = Y(X=0) \qquad = B$$

$$v_0 = y'(x=0) = \quad \omega A \cos(\omega X) - B\omega \sin(\omega X)\Big|_{x=0} =$$

$$= \quad \omega A$$

$$\boxed{y(x) = y_0 \cos(\omega x) + \frac{v_0}{\omega} \sin(\omega X)}$$

Ni dušenja in sistem večno oscilira



$\boxed{v \neq 0}$ pozneje, ko razumemo Eulerjevo enačbo

2.) Taylorjev razvoj okoli točke a

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$\rightarrow f(x) = e^{\alpha X}$ okoli $a=0$

$$f(0) =$$

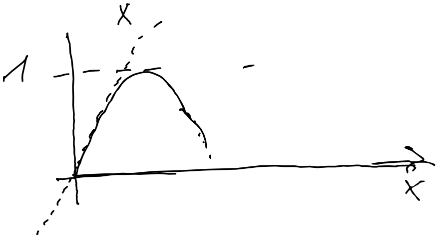
$$\frac{\partial f}{\partial x}\Big|_{x=0} = +\alpha e^{+\alpha x} = +\alpha$$

$$\frac{\partial^2 f}{\partial x^2}\Big|_{x=0} = +\alpha^2 e^{+\alpha x} = \alpha^2$$

$$\frac{\partial^3 f}{\partial x^3}\Big|_{x=0} = \alpha^3 e^{\alpha x} = \alpha^3$$

$$f(x) = 1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!}$$

$\rightarrow f(x) = \sin(x)$



$$f(x=0) = 0$$

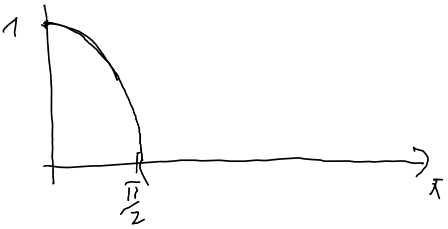
$$f'(x=0) = \cos(x)\Big|_{x=0} = 1$$

$$f''(x=0) = -\sin(x)\Big|_{x=0} = 0$$

$$f^{(3)}(x,=0) = -\cos(x)\Big|_{x=0} = -1$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$\rightarrow f(x) = \cos(x)$



$$f(x=0) = 1$$

$$f'(x=0) = -\sin(x)\Big|_{x=0} = 0$$

$$f''(x=0) = -\cos(x)\Big|_{x=0} = -1$$

$\vdots$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

$\rightarrow f(x) = e^{ix}$

$$e^{ix} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} =$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \dots\right) = \cos x + i \sin x$$

Eulerjeva enačba

$$e^{ix} = \cos x + i\sin x$$

za $x=\pi$ $e^{i\pi} = 1$

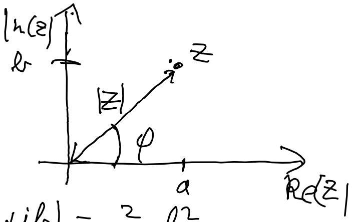
ali

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Pokaži sam

kompleksna števila in Eulerjeva enačba



$Z = a + ib$ $i^2 = -1$

$z^* = a - ib$

Absolutna vrednost $|z|^2 = z^* z = (a-ib)(a+ib) = a^2 + b^2$

$|z| = \sqrt{z^2 z}$

Množenja $z_i = a_i + ib_i$ $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) =$

$= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$

Deljenje $$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(a_1+ib_1)(a_2-ib_2)}{|z_2|^2} = \frac{(a_1a_2+b_1b_2)+i(b_1a_2-a_1b_2)}{a_2^2+b_2^2}$$

Polarni zapis:

$z = |z| e^{i\varphi}$ $\tan(\varphi) = \frac{b}{a}$

Množenje

$$z_1 \cdot z_2 = |z_1| \, |z_2| \, e^{i \varphi_1 + \varphi_2}$$

Deljenje

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \quad e^{i(\varphi_1 - \varphi_2)}$$

Uporaba pri differencialnih enačbah

$$y'' + \alpha y' + \omega^2 y = 0$$

$$y = A e^{\lambda x}$$

$$y' = y \cdot (\lambda)$$

$$y'' = y(+\lambda^2)$$

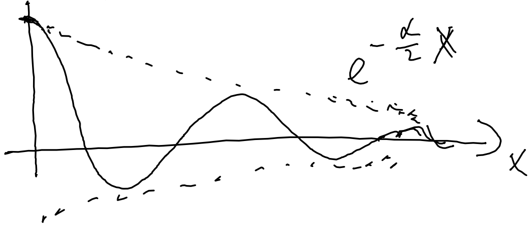
$$(+\lambda^2 + \lambda \alpha + \omega^2) y = 0$$

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\omega^2}}{2}$$

Dušen
harmonski
oscila

$$\underline{\alpha^2 - 4\omega^2 < 0} \qquad \lambda = -\frac{\alpha}{2} \pm i \overbrace{\sqrt{\omega^2 - \left(\frac{\alpha}{2}\right)^2}}^{\tilde{\omega}}$$

$$y = \left(A e^{i \tilde{\omega} x} + B e^{-\tilde{\omega} x}\right) e^{-\frac{\alpha x}{2}}$$



$$\tilde{\omega} = \sqrt{\omega^2 - \left(\frac{\alpha}{2}\right)^2}$$

Kritično dušenje

$\alpha^2 - 4\omega^2 = 0$ $\alpha = 2\omega$

$y(x) = A e^{-2\omega x}$

Adicijski izreki in interferenca

$e^{ix} = \cos(x) + i\sin(x)$

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= (\cos\alpha \cos\beta - \sin\alpha \sin\beta) + i(\cos\alpha \sin\beta + \sin\alpha \cos\beta]$$

Re: $\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

Im: $\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta$

Primeri

a) $\psi_1 = \sin(\omega t)$
$\psi_2 = \sin(\omega t)$

$\psi_1 + \psi_2 = 2\sin(\omega t)$ konstruktivna interferenca

b) $\psi_1 = \sin(\omega t)$
$\psi_2 = \sin(\omega t + \pi)$

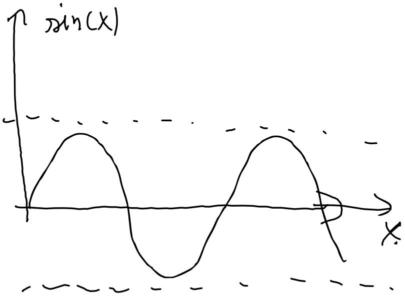
$\psi_1 + \psi_2 = \sin(\omega t) + \sin(\omega t + \pi) =$
$= \sin(\omega t) + \emptyset - \sin(\omega t) = 0$

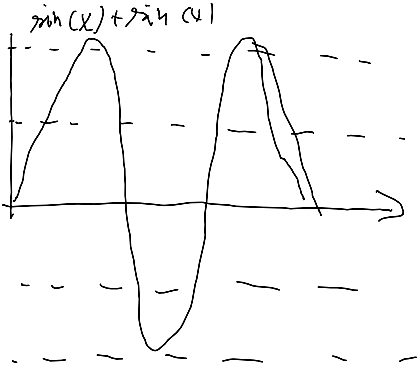
Destruktivno

c) $\sin(\omega t) + \sin(\omega t + \varphi) = 2\sin\left(\omega t + \frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right)$

$\varphi = 0$ - konstruktivno

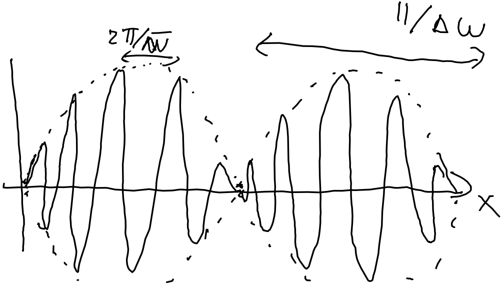
$\varphi = \pi$ - destruktivno





d) Utripanje

$\sin(\omega_1 t) + \sin(\omega_2 t) = 2\sin\left(\frac{\omega_1+\omega_2}{2} t\right)\cos\left(\frac{\omega_1-\omega_2}{2} t\right)$



$\frac{\omega_1+\omega_2}{2} = \bar{\omega}$ $\frac{\omega_1-\omega_2}{2} = \Delta\omega$

Pomembno za grupno in fazno hitrost

Sistem differencialnih enačb

$$\vec{y}\,'(x) + \underline{\underline{A}}\,\vec{y}(x) = 0$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}(x) + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

Nastavek $\vec{y}(x) = \vec{v}\, e^{-\lambda x}$

$$-\underline{\underline{I}}\lambda \vec{v}\, e^{\lambda x} + \underline{\underline{A}}\vec{v} e^{\lambda x} = 0$$

$$e^{\lambda x}\left(-\underline{\underline{I}}\lambda + \underline{\underline{A}}\right)\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0 \quad \text{ali} \quad A\vec{v} = \lambda \vec{v}$$

Problem lastnih vrednosti in lastnih vektorjev:

Transponiranje: $A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$ $\quad A^+ = \left(A^t\right)^*$

$$A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix}$$

Hermitska matrika: $A^+ = A$

Izrek: Hermitske matrike imajo le realne lastne vektorje λ in ortogonalne lastne vektorje

$$\langle \vec{v}_i | \vec{v}_j \rangle = \sum_l [v_i]_l^* [v_j]_l = \delta_{ij}$$

Primer 2x2:

$$A^+ \doteq A \quad \Rightarrow \quad a_{11}, a_{22} \text{ realna}$$
$$a_{12} = a_{21}^*$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\vec{v} = 0 \qquad \det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{12}^* & a_{22}-\lambda \end{pmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{12}^* = 0$$

$$\lambda^2 - \lambda(a_{11}+a_{22}) + a_{11}a_{22} - |a_{12}|^2 = 0$$

Lastne vrednosti:

$$\lambda_{1,2} = \frac{a_{11}+a_{22} \pm \sqrt{(a_{11}+a_{22})^2 + 4|a_{12}|^2 - 4a_{11}a_{22}}}{2}$$
$$= \frac{a_{11}+a_{22} \pm \sqrt{(a_{11}-a_{22})^2 + 4|a_{12}|^2}}{2}$$

Lastni vektorji:

$$D = (a_{11}-a_{22})^2 + 4|a_{12}|^2 \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{\underline{A}}\,\vec{v} = \lambda\vec{v}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(a_{11}+a_{22} \pm D)\begin{pmatrix} x \\ y \end{pmatrix}$$

$\boxed{\text{D.N.}}$ Ne navajah

$$\left\{\begin{array}{l} a_{11}x + a_{12}y = \frac{1}{2}(a_{11}+a_{22} \pm D)\,x \\ a_{12}^*x + a_{22}y = \frac{1}{2}(a_{11}+a_{22} \pm D)\,y \end{array}\right.$$

$$\frac{1}{2}(a_{11} - a_{22} \mp D)\,x + a_{12}y = 0$$
$$a_{12}^*x + \frac{1}{2}(a_{22} - a_{11} \mp D)\,y = 0$$

$$\frac{1}{2}(a_{11}-a_{22}\mp D)\,x + a_{12}\,y = 0$$

$$a_{12}^{*}\,x + \frac{1}{2}(a_{22}-a_{11}\mp D)\,y = 0 \quad / \cdot (a_{11}-a_{22}\mp D) / 2a_{12}^{*}$$

$$\frac{1}{2}\left(a_{11}-a_{22}\mp D\right)x + \frac{1}{2}\,\frac{(a_{22}-a_{11}\mp D)\,(a_{11}-a_{22}\mp D)}{4\,a_{12}^{*}}\,y = 0$$

$$\frac{1}{2}\left(a_{11}-a_{22}\mp D\right)x + \frac{\left[D^2-(a_{22}-a_{11})^2\right]y}{4\,a_{12}^{*}} = 0$$

$$]\; D^2-(a_{22}-a_{11})^2 = |a_{12}|^2\,4$$

$$\frac{1}{2}(a_{11}-a_{22}\mp D)\,x + a_{12}\,y = 0$$

$$y = \frac{a_{22}-a_{11}\pm D}{2\,a_{12}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_1 = \begin{pmatrix} 1 \\ \frac{a_{22}-a_{11}+D}{2a_{12}} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_2 = \begin{pmatrix} 1 \\ \frac{a_{22}-a_{11}-D}{2a_{12}} \end{pmatrix}$$

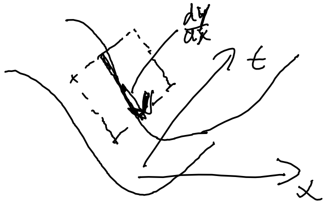
D.N.

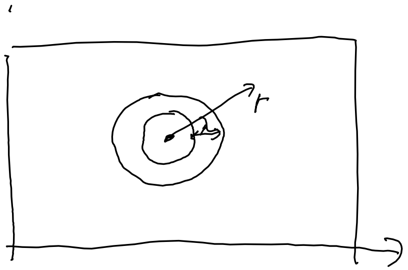
$$D = (a_{11}-a_{22})^2 + 4|a_{12}|^2$$

Parcialne diferencialne enačbe

$y(x,t)$

Parcialni odvod $\frac{\partial}{\partial x} y(x,t) = \lim_{dx \to 0} \frac{y(x+dx,t) - y(x,t)}{dx}$

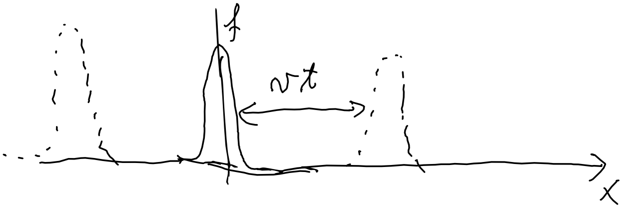




Propagacija vala

$y(x, t=t_0) = f(x)$

$y(x, t=t_1)$



$y(x,t) = f(x-ct)$ desna

$y(x,t) = f(x+ct)$ leva

Valovna enačba

$\xi = x - ct \qquad \eta = x + ct$

$$c^2 \frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial}{\partial (x-ct)} \frac{\partial}{\partial (x+ct)} y = \frac{\partial}{\partial (x-ct)} \left(\frac{\partial y}{\partial x} + \frac{1}{c} \frac{\partial y}{\partial t} \right) = \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2}{\partial t^2} y \frac{1}{c^2} = 0$$

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0 \Leftrightarrow y = F(\xi) + G(\eta) \Rightarrow y(t) = F(x-ct) + G(x+ct)$$

Torej $\psi = \psi_1(x-ct) + \psi_2(x+ct)$ je splošna rešitev valovne enačbe

Nastanek ravnega vala

Nastavek: $\psi(x,t) = \psi_0 \, e^{i(\omega t + kx)}$

$$\frac{\partial}{\partial t^2}\psi = e^{i(\omega t + kx)} \quad (i\omega)^2 \psi_0$$

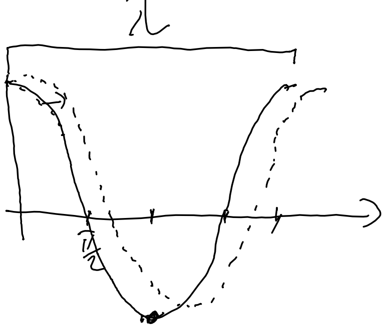
$$\frac{\partial^2}{\partial x^2}\psi = e^{i(\omega t + kx)} \quad (ik)^2 \psi_0$$

$$(-c^2 k^2 + \omega^2) \quad e^{i(\omega t + kx)} \psi_0 = 0$$

Hitrost razširjanja

$$\psi = \psi_0 \, e^{i(\omega t + kx)} \qquad \boxed{\omega = ck}$$

Trenutna slika



$$Re(\psi) = \psi_0 \cos(\omega t + kx)$$

$$k\lambda = 2\pi$$

$k = \frac{2\pi}{\lambda}$ valovna dolžina

$$c = \frac{\omega}{k} = \frac{2\pi \nu \lambda}{2\pi} = \nu\lambda$$

$$\boxed{c = \nu\lambda}$$

Struna

$y(0,t) = y(L,t) = 0$

$$c^2 \frac{\partial^2}{\partial x^2} y(x,t) - \frac{\partial^2}{\partial t^2} y(x,t) = 0$$

$$y(x,t) = \phi(x) \; G(t)$$

$$c^2 \left[\frac{\partial^2}{\partial x^2} \phi(x)\right] G(t) - \left[\frac{\partial^2}{\partial x^2} G(t)\right] \phi(x) = 0$$

$$\frac{\frac{\partial^2}{\partial x^2}\phi(x)}{\phi(x)} = \frac{1}{c^2} \frac{\frac{\partial^2}{\partial x^2} G(t)}{G(t)} = k^2 \qquad \lambda \geq 0$$

Nastavek : $G(t) = A\cos(\omega \cdot t) + B\sin(\omega \cdot t)$

$\phi(x) = C\cos(kx) + D\sin(kx)$

$\frac{\partial^2}{\partial x^2}\phi(x) = k^2\phi$

$\frac{\partial^2}{\partial x^2} G(t) = \omega^2 c^2 G$ $\Rightarrow$ $\underline{\omega = kc}$

R.p. $\phi(0) = 0 \Rightarrow C = 0$

$\phi(L) = 0 = \sin(kL)$ $\qquad kL = n\pi \;\; ; n = 1,2,3,\ldots$

Začetni pogoj $\phi(x) = D\sin\left(\frac{n\pi}{L} x\right)$ $\qquad k_n = \frac{n\pi}{L} \;\; ; n = 1,2,3,\ldots$

$$y(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)\left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right]$$

$$k_n = \frac{n\pi}{L}$$

$$\omega_n = \frac{n\pi c}{L} \rightarrow \nu_n = n\frac{c}{2L}$$

Daljša struna - nižja frekvenca

Začetni pogoji

1.)

$$y(x,0) = \sum_{n=1}^{\infty} \sin(k_n x)\, A_n = g(x)$$

$$A_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = 0$$

če $g(x) = \sin\left(\frac{m\pi x}{L}\right) \rightarrow A_n = \delta_{nm}$

Numerično reševanje diff. enačb

Harmonski oscilator

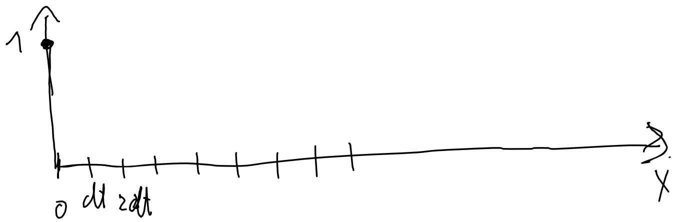
$$m \frac{d^2 y}{d x^2} + k y = 0$$

$y(t=0) = 1$

$y'(t=0) = 0$

$k = 1$

$m = 1$



$$y' = \frac{y(x+dx) - y(x)}{dx}$$

$$y'' = \frac{y'(x+dx) - y(x)}{dx}$$

Eulerjeva metoda

V praksi:
scipy: odeint

$$y_{i+1} = y_i + v_i \Delta t$$

$$v_{i+1} = v_i + \left[-\frac{k x_i}{m} \right] \Delta t$$

Anharmonski oscilator

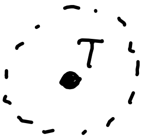
$$m \frac{d^2 y}{d x^2} + k y + k_1 y^3 + k_2 y^5 = 0$$

$$y_{i+1} = y_i + v_i \Delta t$$

$$v_{i+1} = v_i + \left[-\frac{k y_i}{m} + \frac{k_1 y_i^3}{m} + \frac{k_2 y^5}{m} + \dots \right] = 0$$

1.) Stefan-Boltzmann zakon iz Planckovega zakona

$$j = \sigma T^4$$



$$j = \frac{P}{A}$$

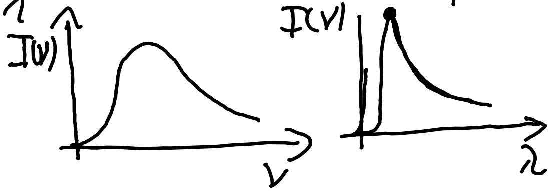
j... gostota energijskega toka

σ... Stefanova konstanta

Spektralna gostota iz Planckovega zakona

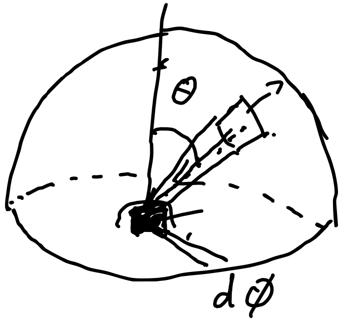
$$I(\nu,T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_BT}}-1}$$

$c = \nu\lambda$ $\lambda^* = \frac{b}{T}$

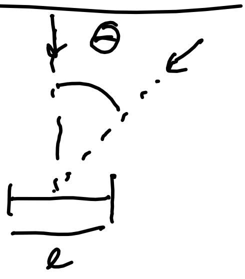


Sferične koordinate

$$d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$



Kosinusni zakon



$$l^* = l \cos\theta$$

$$\frac{P}{A} = \int d\nu\, I(\nu,T) \int_0^{\pi/2} \cos\theta \, \sin(\theta)\, d\theta \int_0^{2\pi} d\varphi =$$

$$\frac{P}{A} = \int_0^{\infty} I(\nu,T)\, d\nu \cdot \underset{①}{\int_0^{\pi/2}} \cos\theta \, \sin\theta \, d\theta \, \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$① \; \frac{1}{2}\int_0^{\pi/2} d\theta \, \sin(2\theta) = \frac{1}{4}\int_0^{\pi} dx \quad \sin(x) =$$

$$x = 2\theta$$

$$= -\frac{\cos(\pi) + \cos(0)}{4} = \frac{1}{2}$$

$$2\cos\theta \sin\theta = \sin(2\theta)$$

$$\frac{P}{A} = \pi \int_0^{\infty} d\nu \, I(\nu,T) = \frac{2\pi h}{c^2} \int_0^{\infty} d\nu \frac{\nu^3}{e^{h\nu/k_BT} - 1} =$$

$$\frac{h\nu}{k_BT} = x$$

$$\frac{h}{k_BT} d\nu = dx$$

$$= \frac{2\pi h}{c^2} \frac{k_BT}{h} \int dx \, \frac{x}{e^x - 1} \left(\frac{k_BT}{h}\right)^3 =$$

$$= \frac{2\pi h}{c^2} \left(\frac{k_BT}{h}\right)^4 \underbrace{\int_0^{\infty} dx \, \frac{x}{e^x - 1}}_{\frac{\pi^4}{15}} =$$

$$= \underbrace{\frac{2\pi^5 k_B^4}{15\, h^3 c^2}}_{\sigma} \, T^4$$

1a) $\varphi_1 = 2\pi\ \nu_1 t$ $\qquad \nu_1 > \nu_2$

$\varphi_2 = 2\pi\ \nu_2\ t$

$\varphi_1 - \varphi_2 = 2\pi t(\nu_1 - \nu_2) = 2\pi$ faza $\quad t = \frac{1}{\nu_1 - \nu_2}$

$= \pi$ antifaza $\quad t = \frac{1}{2}\ \frac{1}{\nu_1 - \nu_2}$

1b) $\varphi_1 = 2\pi\ \nu_1 t + \varphi_0$

$\varphi_2 = 2\pi \nu_2 t$

(faza)

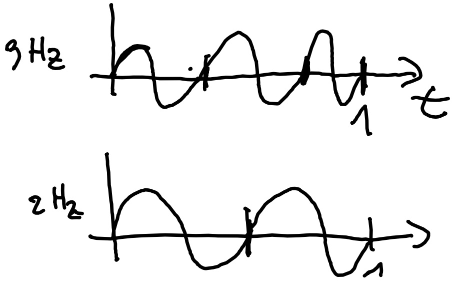
$0 \leq \varphi_0 < \pi \quad \varphi_1 - \varphi_2 = 2\pi(\nu_1 - \nu_2)t + \varphi_0 \quad = 2\pi \qquad t = \frac{1 - \frac{\varphi_0}{2\pi}}{\nu_1 - \nu_2}$

antifaza

$= \pi \qquad t = \frac{-\varphi_0/2\pi}{\nu_1 - \nu_2}$

2.) ν_1, ν_2 → skupna frekvenca

$$\frac{\nu_1}{\nu_2} = \frac{m}{n}$$



$T = \frac{1}{3}$ s $\quad T = \frac{1}{2}$ s $\quad \Big\} \rightarrow 1$ s

$$\frac{\nu_1}{\nu_2} = \frac{m}{n} \qquad T_2 = \frac{1}{\nu_2}$$

$$\frac{\nu_1}{m} = \frac{\nu_2}{n} \qquad T = nT_2$$ ← Daljša perioda

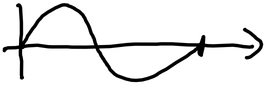
$$\boxed{\nu = \frac{\nu_2}{n}}$$

$$2\pi \nu_1 t = 2\pi \nu_2 t \quad \text{mod } 2\pi$$

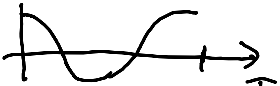
$$\nu_1 t = \nu_2 t \quad \text{mod } 1$$

$\boxed{3.}$ Povprečna hitrost nihanja

$$x = x_0 \sin(\omega t + \varphi_0)$$



$$v = x_0 \omega \cos(\omega t + \varphi_0)$$



$$\bar{v} = x_0 \omega \frac{1}{T}\int_0^T \cos(\omega t + \varphi_0)\, dt = \frac{x_0 \omega}{T} \frac{\sin(\omega t + \varphi_0)}{\omega}\Big|_0^T =$$

$$= \frac{x_0 \omega}{T} \frac{\sin(\overbrace{\omega T}^{2\pi} + \varphi_0) - \sin(\varphi_0)}{\omega} = 0$$

Povprečna hitrost dušenega nihala

$$x=x_0\sin(\Omega t)\,e^{-\alpha t}\qquad \Omega=\sqrt{\omega^2-\alpha^2}$$

$$v(t)=\Omega x_0\cos(\Omega t)\,e^{-\alpha t}+x_0\sin(\Omega t)\,e^{-\alpha t}(-\alpha)=$$

$$=\Omega x_0\cos(\Omega t)\,e^{-\alpha t}+x(t)(-\alpha)$$

$$\bar{v}=\frac{1}{T}\;\Omega x_0\cdot\int_0^T\cos(\Omega t)\,e^{-\alpha t}$$

$$-\alpha\int_0^T x(t)\,dt \qquad\qquad \alpha t=\alpha\frac{x}{\Omega}$$

$$\int_0^T dt\cos(\Omega t)\,e^{-\alpha t}\overset{x=\Omega t}{=}\Omega\int_0^{\Omega T}dx\cos(x)\,e^{-\frac{\alpha x}{\Omega}}$$

$$\int\cos(x)\,e^{-cx}=\frac{e^{-cx}\left[\sin(x)-c\cos(x)\right]}{1+c^2}\qquad C=\frac{\alpha}{\Omega}$$

$$=\Omega\left[e^{-\alpha T}\frac{\overset{\overset{0}{\shortparallel}}{\sin(\Omega T)}-\frac{\alpha}{\Omega}\overset{\overset{1}{\shortparallel}}{\cos(\Omega T)}}{1+\left(\frac{\alpha}{\Omega}\right)^2}+\frac{\frac{\alpha}{\Omega}}{1+\left(\frac{\alpha}{\Omega}\right)^2}\right]=$$

$$\bar{v}=\frac{\Omega\frac{\alpha}{\Omega}}{1+\left(\frac{\alpha}{\Omega}\right)^2}\left(1-e^{-\alpha T}\right)=\frac{\alpha}{1+\left(\frac{\alpha}{\Omega}\right)^2}\left(1-e^{-\alpha T}\right)$$

D.N. Pokaži kaj se zgodi z $\int x(t)\,dt$ (drugi člen)?

10.1 kin in potencial energija dušenega oscilatorja

$$x = x_0 \cos(\omega t + \varphi_0)$$

$$v = x_0 \omega \sin(\omega t + \varphi_0)$$

$$E_{kin} = \frac{m v^2}{2} = \frac{m x_0^2 \omega^2}{2} \sin^2(\omega t + \varphi_0)$$

$$E_{pot} = \frac{k x^2}{2} = \frac{k}{2} x_0^2 \cos^2(\omega t + \varphi_0)$$

$$\omega^2 = \frac{k}{m}$$

$$E_{kin} + E_{pot} = \frac{1}{2} k x_0^2 (\sin^2 + \cos^2) = \frac{1}{2} k x_0^2$$

Dušeno

$$E = \frac{m \dot{x}^2}{2} + \frac{1}{2} k x^2$$

$$\frac{dE}{dt} = m \dot{x} \ddot{x} + k x \dot{x} = \dot{x}(k x + m \ddot{x}) =$$

$$= - \alpha \dot{x}^2 \leq 0$$

$$m \ddot{x} + \alpha \dot{x} + k x = 0$$

Energija se izgublja.

$$- \alpha \dot{x} = m \ddot{x} + k x$$

$$X = X_0 \cos(\Omega t) e^{-\alpha t} \qquad \Omega = \sqrt{\omega^2 - \alpha^2}$$

$$\dot X = X_0 \Omega \sin(\Omega t) e^{-\alpha t} + X_0 \cos(\Omega t) e^{-\alpha t} (-\alpha) =$$

$$= X_0 \Omega \sin(\Omega t) e^{-\alpha t} - \alpha X(t) \qquad \boxed{\omega = \sqrt{\frac{K}{m}} \quad m = \frac{K}{\omega^2}}$$

$$E_{kin} = \frac{m \dot X^2}{2} = \frac{k}{2\omega^2} \Big[X_0^2 \Omega^2 \sin^2(\Omega t) e^{-2\alpha t} + \alpha^2 X^2 - 2\alpha X X_0 \Omega \sin(\Omega t) e^{-\alpha t} \Big]$$

$$= \frac{k}{2\omega^2} \Big[X_0^2 \Omega^2 \sin^2(\Omega t) e^{-2\alpha t} + \alpha^2 X^2 - 2\alpha X X_0 \Omega \sin(\Omega t) e^{-\alpha t} \Big]$$

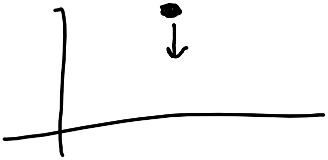
$$= \frac{k}{2\omega^2} e^{-2\alpha t} \Big[X_0^2 \Omega^2 \sin^2(\Omega t) + \alpha^2 X_0^2 \cos^2 \Omega t - 2\alpha X_0^2 \Omega \sin(\Omega t) \cos \Omega t \Big]$$

$$= \frac{k X_0^2}{2\omega^2} e^{-2\alpha t} \big(\Omega \sin(\Omega t) - \alpha \cos(\Omega t) \big)^2$$

$$E_{pot} = \frac{k}{2} X_0^2 \cos^2(\Omega t) \underline{e^{-2\alpha t}}$$

Energija eksponentno pada s časom.

$\boxed{11.}$ Gravitacijsko polje iz sile



$$\vec{F} = -m\vec{\nabla}\phi$$

1D:

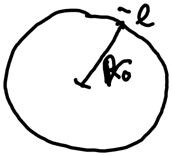
$$F = mg$$

$$F = -\frac{d\phi}{dz} = mg \quad \Rightarrow \quad \phi = -mgz$$

v resnici

$$\phi = -G\frac{m}{r} \qquad \text{za} \qquad \phi(r) = \phi(r_0) + \frac{\partial\phi}{\partial r}(r - r_0) \doteq$$

$$= -\frac{Gm}{r_0} + \frac{Gm}{r_0^2}\underbrace{(r - r_0)}_{\text{linearni odmik}}$$



$$\phi = -\frac{GM}{r_0} + \frac{GM}{r_0^2} z$$

$M_+ = 5{,}9 \cdot 10^{24}\,kg$

$G = 6{,}6 \cdot 10^{-11} \frac{m^3}{kg\,s^2}$

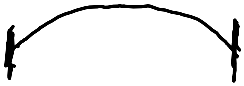
$r_+ = 6{,}3 \cdot 10^{6}\,m$

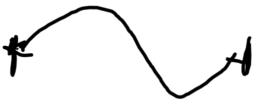
$$\frac{GM}{r_0^2} \doteq \frac{6.6 \cdot 10^{-11}\ 5.9\ 10^{24}}{(6{,}3 \cdot 10^{6})^2} \frac{m^3 \cancel{kg}}{\cancel{kg} s^2 m^2}$$

$$= 9.8 \ \frac{m}{s^2}$$

7. Valovanje napete strune

a)





$\psi(0)=\psi(L)=0$ za vsak t

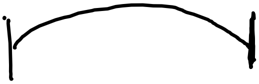
$$c^2 \frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{\partial^2}{\partial t^2}\psi(x,t) = 0$$

$$\psi(x,t) = \phi(x)\, G(t)$$

$$\psi_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right]$$

$\omega = kc$ in $\omega_n = \frac{n\pi}{L} c$

$$\nu_n = n \frac{c}{2L}$$



$\nu = \frac{c}{2L}$

$kL = \pi$

$kL = 2\pi$ $\nu = \frac{c}{L}$

6) Načelo superpozicije

$$c^2 \frac{\partial^2}{\partial x^2} \psi_1 = \frac{\partial^2 \psi_1}{\partial t^2}$$ rol 1

$$c^2 \frac{\partial^2}{\partial x^2} \psi_2 = \frac{\partial^2 \psi_2}{\partial t^2}$$ rol 2

$$\psi = a\psi_1 + b\psi_2$$

$$c^2 \frac{\partial^2}{\partial x^2} \psi = c^2 \frac{\partial^2}{\partial x^2} (a\psi_1 + b\psi_2) = c^2 a \frac{\partial^2}{\partial x^2} \psi_1 + c^2 b \frac{\partial^2}{\partial x^2} \psi_2$$

$$= a \frac{\partial^2 \psi_1}{\partial t^2} + b \frac{\partial^2 \psi_2}{\partial t^2} = \frac{\partial^2}{\partial t^2} (a\psi_1 + b\psi_2) = \frac{\partial^2}{\partial t^2} \psi$$

$$c^2 \frac{\partial^2}{\partial x^2} \psi = \frac{\partial^2}{\partial t^2} \psi$$

Če imamo ~~st~~dve ali več rešitve

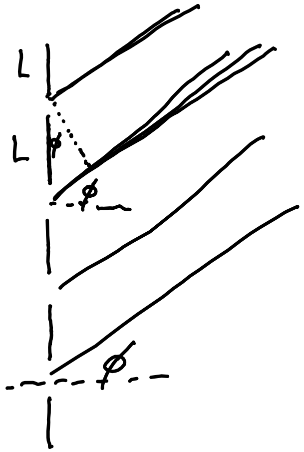
⇒ vsota je enaka kot če bi gledali vsakega posebej.

$\dashrightarrow$ $A \sin(kx) + B \cos(kx)$

$\psi(0) = 0 \Rightarrow B = 0$

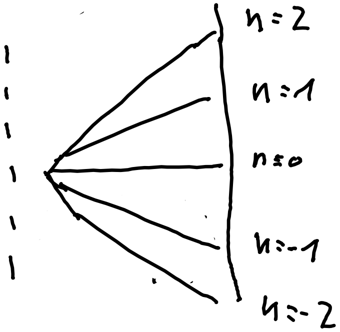
$\frac{\partial \psi}{\partial x}(L) = 0 \qquad A - k \cos(kL) = 0 \qquad kL = \frac{\pi}{2} + n\pi$

9) Uklonska mrežica



$$L \sin\phi = n\lambda$$

$$\sin\phi = \frac{n\lambda}{L}$$



12.) Fotoefekt in izstopno delo

$\Phi = 4$ kakšna je valovna dolžina fotona

$$\underbrace{h\nu}_{\text{Energija fotona}} = \Phi + E_{kin}$$

$$c = \lambda\nu$$

$$hc = 1.2\ \text{eV}\mu\text{m}$$

Pogoj $E_{kin} = 0$ $\quad h\nu = \Phi$

$$\frac{hc}{\lambda} = \phi$$

$$\lambda = \frac{hc}{\phi} = \frac{1.2\text{eV}\mu\text{m}}{4\text{eV}} = 0.3\ \mu\text{m} = 300\ \text{nm}$$

$\boxed{\text{UV}}$

13.) Stefan - Boltzman zakon

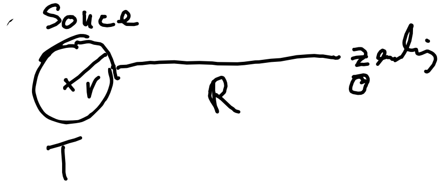
Kirchhoffov zakon o termičnem sevanju

- Absorbcija in sevanje sta povezana

$$\underbrace{I(\lambda)}_{\text{gostota sevalnega toka}} = I_{črno}(\lambda)\,[1-\underbrace{a(\lambda)}_{\text{albedo}}]$$

albedo → 0 črno
→ 1 belo idealno zrcalo

Razmerje med sevanjem in absorbcijo je odvisno le od $\nu(\lambda)$ in T.



$r = 7 \cdot 10^8$ m

$R = 0.15\ 10^{12}$ m

$\sigma = 5.6 \cdot 10^{-8} \frac{W}{m^2 K^4}$

$T = 5800$ K

$$j = \sigma T^4$$

$$\underbrace{j_z}_{\text{Solarna konstanta}} = j\,\frac{\cancel{4\pi} r^2}{\cancel{4\pi} R^2} = \underbrace{\sigma T^4}_{6.4\cdot 10^7 \frac{W}{m^2}}\,\underbrace{\frac{r^2}{R^2}}_{2\cdot 10^{-5}} = 1300\ \frac{W}{m^2}$$

Izmerjeno $\bar{j}_z \approx 1360\ \frac{W}{m^2}$

$E_{sev} = E_{izse}$

$\cancel{4}\,\cancel{\pi}\,\cancel{r_z^2}\; j_z = \sigma T_z^4 \; 4\pi \cancel{r_z^2} \cdot \cancel{4}$

$$T_z = \left(\frac{j_z}{2\sigma}\right)^{\frac{1}{4}} = 300\ K \sim 58\ °C$$

Albedo ne določa T_z v ravnovesju
pomemben za hitrost vračanja v ravnovesje

14). Koliko kalorij porabimo v 1h zaradi sevanja ($T = 293\ K$ in $T = 273\ K$)

$1\ J \sim 0{,}24\ cal$ in $1g$ čokolade $= 5 \cdot 10^3\ cal$

Črno telo:

$P = j\, S = \sigma T^4 \cdot S$ $\quad S = 2\ m^2$

$$= 5{,}6 \cdot 10^{-8} \frac{W}{m^2 K^4} (293\ k)^4\, 2 m^2 = 825\ W$$

$W = \cancel{8}825 \frac{J}{s} \cdot 3600\ s = 3 \cdot 10^6\ J \rightarrow 700\ kcal$

proti

$1g$ čokolade $\rightarrow 5\ kal$
$100g$ čokolade $\rightarrow 500\ kcal$

15.) Planckov zakon

$$I(\nu,T)=\frac{2h\nu^3}{c^2}\,\frac{1}{e^{\frac{h\nu}{k_BT}}-1}$$

ali

$$I(\lambda,T)=\frac{2hc^2}{\lambda^5}\,\frac{1}{e^{hc/k_B\lambda T}-1}$$

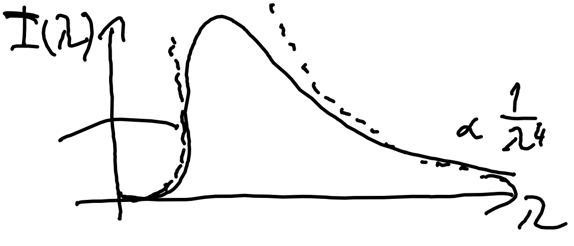
$\underline{\underline{\lambda\to 0}}$

$$I(\lambda,T)\approx\frac{2hc^2}{\lambda^5}\,e^{-\frac{hc}{k_B\lambda T}}$$ Wienova aproksimacija

$\lambda\gg\infty$

$$\frac{1}{e^x-1}\approx\frac{1}{1+x-1}\approx\frac{1}{x}$$

$$I(\lambda,T)=\frac{2\cancel{h}c^{\cancel{2}}}{\lambda^{\cancel{5}4}}\,\frac{k_B\cancel{\lambda}T}{\cancel{h}\cancel{c}}=\frac{2ck_BT}{\lambda^4}$$ Rayleigh-Jeans model



16.) Pospešek vesoljske ladje na svetlobni tok iz telesa $T=10\,k$

$j = \sigma T^4$ $\qquad F = ma = \frac{d}{dt}(mv) = \frac{dp}{dt}$

$E = p \cdot c$

Moment

tlak $\quad p = \frac{F}{S} = \frac{p}{t}\,\frac{1}{S} = \frac{E}{c}\,\frac{1}{t}\,\frac{1}{S} = \frac{j}{c} = \frac{\sigma T^4}{c}$

$$\frac{F}{S} = p = \frac{\sigma T^4}{c}$$

$$ma = \frac{\sigma T^4}{c} S \qquad \underline{\underline{a = \frac{\sigma T^4 S}{c m}}}$$

$$\rightarrow \tilde{a} = 2\,\frac{\sigma T^4 S}{c m}$$

$\Delta p = 2 p_0$

$\boxed{17+18}$ E, p, L

$$E = h\nu = h\frac{c}{\lambda} \Rightarrow \lambda = \frac{hc}{E}$$

$$E = pc$$

$$p = \frac{E}{c} = \frac{hc}{\lambda c} = \frac{h}{\lambda}$$

Vidna svetl. 400 - 700 nm

$$E = \frac{hc}{\lambda} = \frac{1.2\ eV\,\mu m}{\lambda} = \begin{cases} \rightarrow 0.48\ eV\ (400\,nm) \\ \rightarrow 0.84\ eV\ (700\,nm) \end{cases}$$

Ne $\phi = 2\ eV$ je red pruzem

$\boxed{19}$ $\lambda = 635\ nm$

$$E = \frac{hc}{\lambda} = \frac{1.2\ eV\,\mu m}{0.635\,\mu m} = 1.95\ eV$$

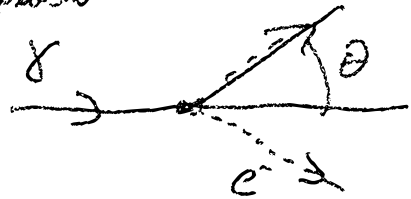
$$\nu = \frac{c}{\lambda} = \frac{3\cdot10^{8}\ m/s}{0.63\ 10^{-6}\ m} = 4.7\cdot10^{14}\,Hz = 472\ Thz$$

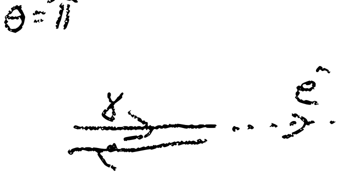
$$p = \frac{E}{c} = \frac{h}{\lambda} = \frac{6.6\cdot10^{-34}\,Js}{0.635\cdot10^{-6}\,m} = 1.0\cdot10^{-27}\ kg\,m/s$$

$$N_\gamma = \frac{P\cdot t}{E} = \frac{1\cdot10^{-3}\ J/\cancel{s}\cdot10^{-11}\cdot \cancel{s}}{1.95\cdot1.6\cdot10^{-19}\,J} = \frac{10^{-14}\,\cancel{J}}{1.95\cdot1.6\cdot10^{-19}\,\cancel{J}} = 32000$$

Comptonovo sipanje (sipanje γ na e^-)

Splošno





Energija

$$E_0 = h\nu_0 = h\frac{c}{\lambda_0}$$

$$E_1 = h\nu_1 + \frac{1}{2}m_e v^2 = h\frac{c}{\lambda_1} + \frac{1}{2}m_e v^2$$

Moment $(\theta = \pi)$

$$\frac{E_0}{c} = -\frac{E_1}{c} + m_e v \qquad v = \frac{E_0+E_1}{m_e c}$$

$$hc\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_1}\right) = \frac{1}{2}m_e\left(\frac{E_0+E_1}{m_e c}\right)^2$$

$$hc\,\frac{\lambda_1-\lambda_0}{\lambda_0\lambda_1} = \frac{m_e}{2}\,\frac{(E_0+E_1)^2}{m_e^2c^2}$$

Zelo majhna sprememba valovne dolžine $\quad E_0+E_1 = \frac{hc}{\lambda}\cdot 2$

$\lambda_0 \sim \lambda_1 \sim \lambda$

$$hc\cdot\frac{\Delta\lambda}{\lambda^2} = \frac{\cancel{m_e}}{2}\,\frac{h^2\cancel{c^2}\cdot 4^2}{\lambda^2 m_e^{\cancel{2}}\cancel{c^2}} = \frac{2h^2}{\cancel{\lambda^2}\, m_e}$$

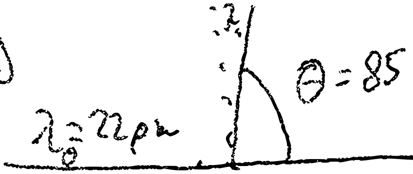
$$\Delta\lambda = 2\underbrace{\boxed{\frac{h}{m_e c}}}_{\lambda_c''}$$

Splošno

$\Delta\lambda = \lambda_c\,(1-\cos\theta)$

$\lambda_c = 2.4\ \text{pm}$

$\boxed{22}$



$$\theta = \frac{85}{180} \cdot \pi$$

$$\Delta\lambda = \lambda_c(1 - \cos\theta) = 2.4pm \cdot (1 - 0.087) =$$
$$= 2.19\,pm$$

$$\lambda_1 = (22 + 2.19)\,pm = 24.19\,pm$$

$\boxed{24}$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\rightarrow A^{\dagger} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{*} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$\rightarrow (A^{\dagger})^{\dagger} = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}^{*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\rightarrow (A^{-1})^{\dagger} \overset{?}{=} (A^{\dagger})^{-1}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \det A = ad - bc$$

$$(A^{-1})^{\dagger} = \frac{1}{a^*d^* - b^*c^*} \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}$$

$$(A^{\dagger})^{-1} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}^{-1} = \frac{1}{a^*d^* - b^*c^*} \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}$$

$$\sim (A+B)^{\dagger} = A^{\dagger} + B^{\dagger} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$(A+B)^{\dagger} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}^{\dagger} = \begin{pmatrix} a_{11}^*+b_{11}^* & a_{21}^*+b_{11}^* \\ a_{12}^*+b_{12}^* & a_{22}^*+b_{22}^* \end{pmatrix}$$

$$A^{\dagger}+B^{\dagger} =$$

$$\sim (A\,B)^{\dagger} = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right]^{\dagger} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & \cdots \\ \cdots & \ddots \end{bmatrix}^{\dagger}$$

$$= \begin{bmatrix} a_{11}^* b_{11}^* + a_{12}^* b_{21} & a_{21}^* b_{11}^* + a_{22}^* b_{21}^* \\ \cdots & \cdots \end{bmatrix}$$

$$B^{\dagger} A^{\dagger} = \begin{bmatrix} b_{11}^* & b_{21}^* \\ b_{12}^* & b_{22}^* \end{bmatrix} \begin{bmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{bmatrix} =$$

$$\begin{bmatrix} b_{11}^* a_{11}^* + b_{21}^* a_{12}^* & b_{11}^* a_{21}^* + b_{21}^* a_{22}^* \\ \cdots & \cdots \end{bmatrix}$$

Kroneckerjev produkt

$$A \otimes B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =$$

kroneckerjev produkt

$$A \otimes B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =$$

$$= \left(\begin{array}{c|c} a_{11}B & a_{12}B \\ \hline a_{21}B & a_{22}B \end{array}\right) = \left(\begin{array}{cc|c} a_{11}b_{11} & a_{11}b_{12} & \\ a_{11}b_{21} & a_{11}b_{22} & \\ \hline a_{21}b_{11} & a_{21}b_{12} & \\ a_{21}b_{21} & a_{21}b_{22} & \end{array}\right)$$

$$(A \otimes B)^\dagger = \left(\begin{array}{c|c} a_{11}^* B^\dagger & a_{21}^* B^\dagger \\ \hline a_{12}^* B^\dagger & a_{22}^* B^\dagger \end{array}\right)$$

$$A^\dagger \otimes B^\dagger = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \otimes B = \begin{pmatrix} a_{11}^* B & a_{21}^* B \\ a_{12}^* B & a_{22}^* B \end{pmatrix}$$

4) $c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\qquad \vec{c} = 1\,\hat{e}_1 + 2\,\hat{e}_2$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\tilde{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \tilde{e}_1 = (e_1 + e_2)\frac{1}{\sqrt{2}}$ $\Rightarrow$ $\frac{\tilde{e}_1 + \tilde{e}_2}{\sqrt{2}} = e_1$

$\tilde{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad \hat{\tilde{e}}_2 = (e_1 - e_2)\frac{1}{\sqrt{2}}$ $\qquad \frac{\tilde{e}_1 - \tilde{e}_2}{\sqrt{2}} = e_2$

$$\vec{c} = (\hat{e}_1 + \tilde{e}_2)\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}(\tilde{e}_1 - \tilde{e}_2) = \tilde{e}_1 \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tilde{e}_2$$

$\boxed{25}$ Paulijeve matrike

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1) $[A,B] = AB - BA$

$\{A,B\} = AB + BA$

$[B,A] = BA - AB = - [A,B]$

2) $$[\sigma_x, \sigma_y] = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3) $[\sigma_i, \sigma_j] = 2i\, \varepsilon_{ijk} \sigma_k$

$$\varepsilon_{ijk} = \begin{cases} 1 & \overset{i\ j\ k}{(1\ 2\ 3)}, (2\ 3\ 1) \\ -1 & (3\ 2\ 1), (1\ 3\ 2), (2\ 1\ 3) \\ 0 & \text{sicer} \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$[\sigma_x, \sigma_y] = 2i\, \varepsilon_{xyz} \sigma_z = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4) $\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_i^2 = \mathbb{1}$

5) Hermitske $\sigma_i^\dagger = \sigma_i \qquad \sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

6) Antikomutator $\{\sigma_i, \sigma_j\} = 2\delta_{ij} \mathbb{1}$

6) Antikommutator

$$\{\sigma_x,\sigma_y\} = \sigma_x\sigma_y + \sigma_y\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \begin{pmatrix}i&0\\0&-i\end{pmatrix} + \begin{pmatrix}-i&0\\0&i\end{pmatrix} = 0$$

$$\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbb{1} \qquad \delta_{ij} = \begin{cases}1; & i=j\\0, & i\neq j\end{cases}$$

7.) $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$

8) $\mathrm{Tr}(\sigma_i) = 0$

$\det(\sigma_i) = -1$

$\boxed{26}$ $[A,B]^\dagger = (AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$

$$M^\dagger = \left(i[A,B]\right)^\dagger = (-i)(AB-BA)^\dagger = (-i)\left(B^\dagger A^\dagger - A^\dagger B^\dagger\right) =$$

$$= -i(BA-AB) = i(AB-BA) = i[A,B] = M$$

29 ~~a~~ a) Lastna stanja Paulijevih matrik

b) Lastna stanja v Diracovem zapisu (braket)

c) Lastna stanja σ_x z lastnimi stanj σ_z

a) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$ $\lambda^2 - 1 = 0$

$\lambda_{0,1} = \pm 1$

vektor za $\lambda = 1$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$ $a - b = 0$, $a = b$ $v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$ $a = -b$ $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) $\sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\det \begin{pmatrix} -\lambda & i \\ -i & -\lambda \end{pmatrix} = 0$ $\lambda^2 - 1 = 0$

$\lambda_{0,1} = \pm 1$

$\lambda = 1$ $\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$ $a = bi$, $-a + bi = 0$ $v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda = -1$ $\begin{pmatrix} +1 & i \\ -i & +1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$ $a + ib = 0$, $a = -ib$ $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

c) $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \lambda_{0,1} = \pm 1$

$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) Diracov zapis

z: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle$

x: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) = |+\rangle$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) = |-\rangle$

c) Izrazi lastna stanja σ_x z σ_z

$$\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) = \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle + |\uparrow\rangle - |\downarrow\rangle\right) = |\uparrow\rangle$$

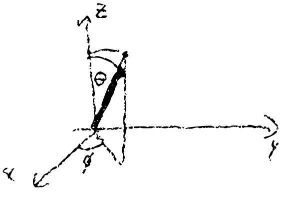
$$\frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) = \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle - \left(|\uparrow\rangle - |\downarrow\rangle\right)\right) = |\downarrow\rangle$$

94.) Blochova sfera in kubit (dvo-nivojski sistem)

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$; $|\alpha|^2 + |\beta|^2 = 1$

$\alpha, \beta \in \mathbb{C} \Rightarrow$ 4 realne št. ⟨ odpade normalizacij / samo rel. faza merljiva

klasična



$x = \sin\theta \sin\phi$

$y = \sin\theta \cos\phi$

$z = \cos\theta$

$x + iy = \sin\theta\, e^{i\phi}$

Parametriziraj : $\alpha = \cos\left(\frac{\theta}{2}\right)$

$\beta = e^{i\phi} \sin\left(\frac{\theta}{2}\right)$

$$\psi = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\theta/2\right)|1\rangle$$

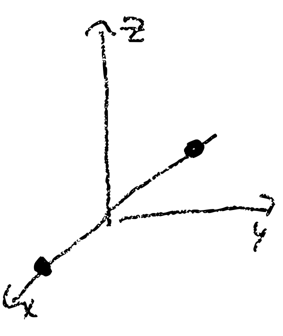
Pozor : Iz ene meritve dobimo ~~en~~ samo dva možna izida → veliko meritev, ala nij α, β

Lastna stanja Paulijevi

σ_x: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \vec{v}_{0,1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$ $\begin{matrix} |0\rangle \\ |1\rangle \end{matrix}$

$v_{0,1}$: $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{2}$

$e^{i\phi}\sin\left(\frac{\theta}{2}\right) = \pm\frac{1}{\sqrt{2}} \quad \varphi = 0, \pi$

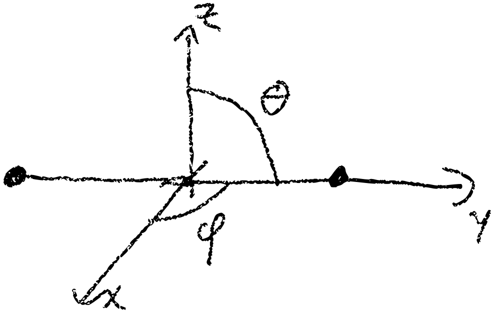


$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \vec{v}_{0,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

$$\frac{1}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right)$$
$$\pm i \frac{1}{\sqrt{2}} = \sin\left(\frac{\theta}{2}\right) e^{i\phi} \quad \Rightarrow \quad \theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{2}, -\frac{\pi}{2}$$

ker $e^{i\frac{\pi}{2}} = i$

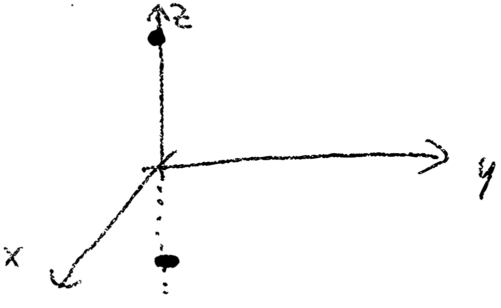
$e^{-i\frac{\pi}{2}} = -i$



σ_z: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{v} = \lambda \vec{v}$ že v diagonalni obliki

$\vec{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \theta = 0$; severni pol

$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \theta = \pi$ južni pol



30.) $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

a) Normiraj stanj

$$\langle\psi|\psi\rangle = (\alpha^*\langle\uparrow| + \beta^*\langle\downarrow|)\,(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)$$

$$= |\alpha|^2 + |\beta|^2$$

$$|\varphi\rangle = \frac{1}{\sqrt{\langle\psi|\psi\rangle}}\,|\psi\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\,(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)$$

b) $p_\uparrow = |\langle\uparrow|\varphi\rangle|^2 = \frac{\alpha^2}{|\alpha|^2+|\beta|^2}$

$p_\downarrow = |\langle\downarrow|\varphi\rangle|^2 = \frac{|\beta|^2}{|\alpha|^2+|\beta|^2}$

$p_\uparrow + p_\downarrow = 1$

c) $|\uparrow\rangle$

$$p_+ = |\langle+|\uparrow\rangle|^2 = \left|\frac{1}{\sqrt2}(\langle\uparrow| + \langle\downarrow|)\,|\uparrow\rangle\right|^2 =$$

$$= \frac{1}{2}$$

$$p_- = |\langle-|\uparrow\rangle|^2 = \left|\frac{1}{\sqrt2}(\langle\uparrow| - \langle\downarrow|)\,|\uparrow\rangle\right|^2 =$$

$$= \frac{1}{2}$$

d) $|\psi_1\rangle,\ |\psi_2\rangle = e^{i\phi}|\psi_1\rangle$

$$|\langle i|\psi_2\rangle|^2 = |\langle i|e^{i\phi}\psi_1\rangle|^2 = |e^{i\phi}|^2 \cdot |\langle i|\psi_1\rangle|^2 = |\langle i|\psi_2\rangle|^2$$

26a) $[A,B]^\dagger = [B^\dagger, A^\dagger]$

b) Pokaži, da je $\left(i[A,B]\right)^\dagger = i[A,B]$

če $A^\dagger = A$
$B^\dagger = B$

$$[A,B]^\dagger = (AB - BA)^\dagger \underset{\uparrow}{=} B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$$

? $(AB)^\dagger = B^\dagger A^\dagger$

$$C = i[A,B]$$

$$C^\dagger = (-i)\left([A,B]\right)^\dagger = -i\,[B^\dagger, A^\dagger] \underset{\uparrow}{=} i\,[A^\dagger, B^\dagger] = C$$

? $[A,B] = -[B,A]$

31) Projektorji

Ket $|j\rangle = \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{bmatrix}$ Bra $\langle i| = \begin{bmatrix} i_1^* & i_2^* & \cdots & i_n^* \end{bmatrix}$

$$\langle i|j\rangle = \begin{bmatrix} i_1^* & i_2^* & \cdots \end{bmatrix} \begin{bmatrix} j_1 \\ \vdots \\ j_n \end{bmatrix} = \sum_{k=1}^{n} i_k^* j_k$$

Meritev + Bornovo pravilo

$$p_j = |\langle j|\psi\rangle|^2$$

Pogoji za projektor: $\hat{P} = \hat{P}^2$ in $\hat{P}^+ = P$

$$P = \sum_{i=1}^{N} |i\rangle\langle i|$$

ortogonalni projektor ?
za $\langle i|j\rangle = \delta_{ij}$

$$P^2 = \left(\sum_i |i\rangle\langle i|\right)\left(\sum_j |j\rangle\langle j|\right) =$$

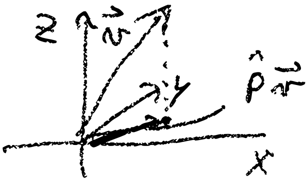
$$= \sum_{i,j} |i\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} \langle j| = \sum_{i}^{N} |i\rangle\langle i|$$

$$P^+ = \left(\sum_i^N |i\rangle\langle i|\right)^+ = \sum_i |i\rangle\langle i| = P$$

$$\left.\begin{array}{l} |i\rangle^+ = \langle i| \\ \langle j|^+ = |j\rangle \end{array}\right]$$

$$Q = 1 - P = \sum_{i=N+1}^{d} |i\rangle\langle i|$$

$$Q^2 = \sum_{\substack{i=N+1 \\ j=N+1}}^{d} |i\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} \langle j| = Q$$



Primer : 3D : $\hat{P}_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\hat{P}_{xy}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$P_z = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P_z^2 = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & \textcircled{0} \end{bmatrix} = P_z$$

$$P_z\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha x + y \\ 0 \end{bmatrix}$$ ← Poševna projekcija

32.) $\hat{A} \rightarrow$ matriko (zapišemo v neki bazi)

$$\langle e_i | \hat{A} | e_j \rangle = \underline{A_{ij}}$$

matrika

↑ baza ↑ operator

Polna baza $\mathbb{1} = \sum_i |e_i\rangle\langle e_i| = \sum_i |f_i\rangle\langle f_i|$

$$|f_i\rangle = \mathbb{1}|f_i\rangle = \sum_j |e_j\rangle \underbrace{\langle e_j | f_i\rangle}_{U_{ji}} = \sum_j U_{ji}|e_j\rangle$$

$$\langle f_i| = \sum_k U_{ki}^* \langle e_k|$$

$$\langle f_i|A|f_j\rangle = \sum_{k,l} U_{ki}^* \langle e_k|\hat{A}|e_l\rangle U_{lj} =$$

$$\langle f_i|A|f_j\rangle = \sum_{k,l} U_{ki} \langle e_k|\hat{A}|e_l\rangle U_{lj} =$$

$$= \sum_{k,l} \underset{\substack{\| \\ (U^\dagger)_{ik}}}{U^*_{ki}} \, A_{kl} U_{lj} = (U^\dagger)_{ik} A_{kl} U_{lj}$$

$$\underset{\substack{\uparrow \\ \text{Nova} \\ \text{baza}}}{\tilde{A}} = U^\dagger \underset{\substack{\uparrow \\ \text{stara baza}}}{A} U$$

27.) Izrazi operator σ_z u bazi $|+\rangle$

$|-\rangle$

$\sigma_x |\pm\rangle = \lambda |\pm\rangle$

$$\tilde{A}_{ij} = (U^{\dagger})_{i\ell} A_{\ell k} U_{kj}$$

$e_i = \begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases}$

$f_i = \begin{cases} |+\rangle \\ |-\rangle \end{cases}$

$$U_{ij} = \langle e_i | f_j \rangle$$

$$\begin{array}{cc} & \begin{array}{cc} |+\rangle & |-\rangle \end{array} \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} & \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \end{array}$$

$$\langle \uparrow | + \rangle = \langle \uparrow | \, (|\uparrow\rangle + |\downarrow\rangle) \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\langle \downarrow | - \rangle = \langle \downarrow | \, (|\uparrow\rangle - |\downarrow\rangle) \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\tilde{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Kolokvij

27.] Izraži operator σ_z v bazi $|+\rangle$
$|-\rangle$

$$\sigma_x |\pm\rangle = \lambda |\pm\rangle$$

$$\tilde{A}_{ij} = U^{\dagger}_{il} A_{lk} U_{kl}$$

$$e_i = \begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases}$$

$$f_j = \begin{cases} |+\rangle \\ |-\rangle \end{cases}$$

$$U_{ij} = \langle e_i | f_j \rangle$$

$$= \begin{array}{c|cc} & |+\rangle & |-\rangle \\ \hline \uparrow & 1/\sqrt{2} & 1/\sqrt{2} \\ \downarrow & 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{array}$$

$$\langle \uparrow | + \rangle = \frac{1}{\sqrt{2}} \langle \uparrow | \left(|\uparrow\rangle + |\downarrow\rangle \right) = \frac{1}{\sqrt{2}}$$

$$\langle \downarrow | - \rangle = \frac{1}{\sqrt{2}} \langle \downarrow | \left(|\uparrow\rangle - |\downarrow\rangle \right) = -\frac{1}{\sqrt{2}}$$

$$\tilde{\sigma}_z = U^{\dagger} \sigma_z U$$

$$\tilde{\sigma}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Rotacija operatorja

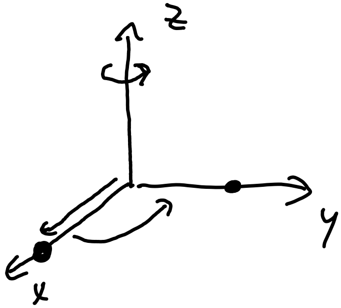
* Rotacija stanja

<u>Splošno</u>

$$U|\varphi\rangle = |\psi\rangle$$

Rotacija spinor $U = e^{-i\frac{\theta^z\sigma^z}{2}} = I\cos\left(\frac{\theta^z}{2}\right) + i\sigma^z\sin\left(\frac{\theta^z}{2}\right)$

$\theta = \frac{\pi}{2}$



$$U|+\rangle = \left(I\frac{\sqrt{2}}{2} - i\,\sigma^z\frac{\sqrt{2}}{2}\right)\cancel{|}+\rangle =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1-i & 0\\ 0 & 1+i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix}1-i\\ 1+i\end{bmatrix} = \frac{1}{2}(1-i)\begin{bmatrix}1\\ \frac{1+i}{1-i}\end{bmatrix} = \frac{1}{4}[1-i]\begin{bmatrix}1\\ i\end{bmatrix}$$

$$\left.\right] \frac{1+i}{1-i} = \frac{(1+i)(1-i)}{|1-i|^2} = \frac{1-1+2i}{2} = i$$

$$= C\underbrace{\begin{bmatrix}1\\ i\end{bmatrix}}_{}$$

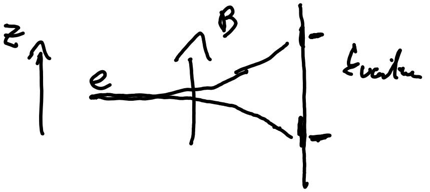
a lastno stanje

$\sigma_y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$

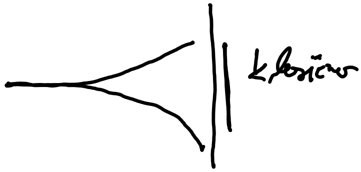
$\boxed{33.}$ Stern-Gerlachov eksperiment

Potencial energije magnetnega moment.

$$U = -\vec{\mu}\vec{B} = -\mu_z B_z$$



$$F_z = -\frac{dU}{dz} = \mu_z \frac{dB_z}{dz}$$



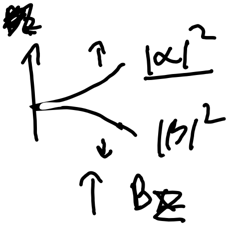
Stanje prej $\psi = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

a) en aparat z gradientom v smeri z $\odot B_z$

$$|\langle\uparrow|\psi\rangle|^2 = |\alpha|^2$$

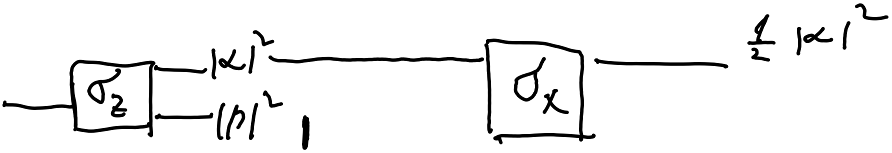
b) dva aparata z gradientom z

sledi v smeri x

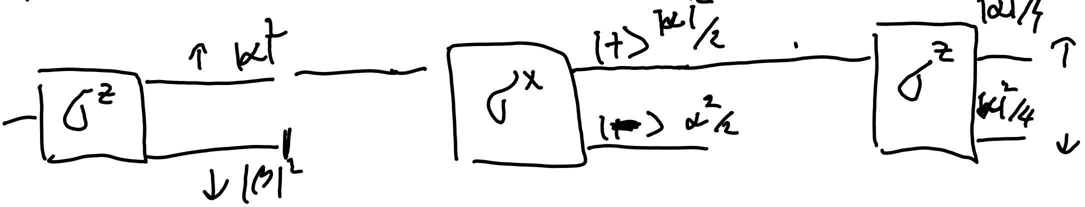


$$\langle +|\uparrow\rangle = \frac{1}{\sqrt{2}}(\langle\uparrow|+\langle\downarrow|)|\uparrow\rangle = \frac{1}{\sqrt{2}}$$

$$\psi' = |\uparrow\rangle$$



c) z - x - z



$$\langle\uparrow|+\rangle = \frac{1}{\sqrt{2}}\langle\uparrow|\left(|\uparrow\rangle+|\downarrow\rangle\right) = \frac{1}{\sqrt{2}}$$

$$|\langle\uparrow|+\rangle|^2 = \frac{1}{2}$$

Valovne lastnosti snovi (dualnost delec-valovanje)

20.) Izračunaj de Broglijevo valovno dolžino

$m = 60 g$ $v = 30 m/s$

a) $\lambda = \frac{h}{p}$ $p = m \cdot v = 0.06 kg \, 30 \frac{m}{s} = 1.8 kg \frac{m}{s}$

$$\lambda = \frac{h}{p} = \frac{6.626 \cdot 10^{-34} kg\, m^2/s}{1.8 kg\, m/s} = 3.6 \cdot 10^{-34} m$$

b) Relativistični delec $E^2=(mc^2)^2+p^2c^2$ ali $E=\gamma mc^2$

$p=\gamma m v$

$v=\frac{c}{2}$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \approx 1.15$$

Klasično

$$p^c = m\cdot v = 9.11\cdot10^{-31}\,kg\,\frac{1}{2}\,3\cdot10^{8}\,\frac{m}{s} = 1.36\cdot10^{-22}\,kg\,\frac{m}{s}$$

$$\lambda_c = \frac{h}{p^c} = \frac{6.62\cdot10^{-34}\,kg\,m^2/s}{1.36\cdot10^{-22}\,kg\,m/s} = 4.8\cdot10^{-12}\,m$$

$\sim pm$

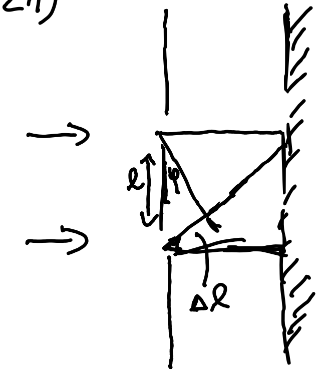
Relativist $p^r=p^c\cdot\gamma \sim 15\%$ več

$$\lambda_R = 4.8\cdot10^{-12}\,m/1.15 = 4.17\cdot10^{-12}\,m$$

? Zakaj zanemarim mirovno ~~mas~~ maso?

Double-slit eksperiment

21)



Kvantno (brez časovne odvisnosti)

$$I=\left|\sin(kx)+\sin(k(x+\Delta l))\right|^2 =$$

$$\sin(a)+\sin(b)=2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\left|2\sin(kx+k\Delta l)\cdot\cos(k\Delta l)\right|^2=$$

$$=4\sin^2(k(x+\Delta l))\cos^2(k\Delta l)$$

Kvantno (časovno odvisno)

$$I = |\sin(kx-\omega t)+\sin(k(x+\Delta l)-\omega t)|^2$$

$$\sin(a)+\sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

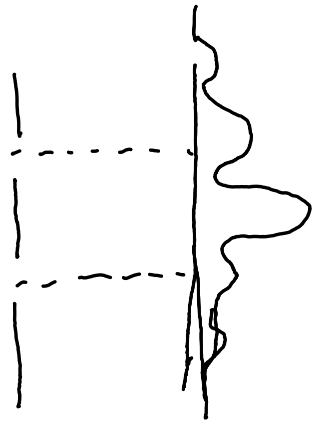
$$\left|2\sin(kx+k\Delta l-\omega t)\cos\left(\frac{k\Delta l}{2}\right)\right|^2 =$$

$$\overline{I(t)} = \frac{1}{T}\int_0^T I(t)\,dt = 4\cos^2\left(\frac{k\Delta l}{2}\right) \cdot \underbrace{\frac{1}{T}\int_0^T \sin^2(kx+k\Delta l-\omega t)\,dt}_{\frac{1}{2}}$$

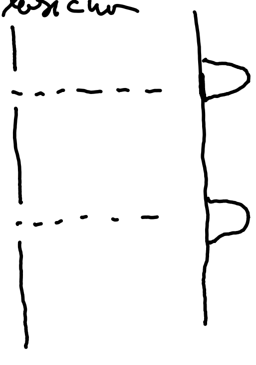
$$\overline{I}(t) = 2\cos^2\left(\frac{k\Delta l}{2}\right) = 2\cos^2\left(\frac{2\pi}{\lambda}\frac{\Delta l}{2}\right) =$$

$$\propto 2\cos^2\left(\frac{\pi l}{\lambda}\sin\varphi\right)$$

kvantno



klasično



k 36.) Ponovi vektorske in matricne operacije

$$\vec{v}\cdot\vec{w} = v_1^* w_1 + v_2 w_2 + \dots = \langle v|w\rangle$$

$$\vec{v}\otimes\vec{w} = |v\rangle \otimes \langle w| = \begin{pmatrix} v_1 w_1^* & v_1 w_2^* & \cdots \\ v_2 & v_2 w_2^* & \cdots \\ \vdots & \vdots & \ddots \\ v_n & & \end{pmatrix}$$

$$A B = A_{ij} B_{jk}$$

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots \\ a_{21} B & \ddots & \\ \vdots & & \end{pmatrix}$$

$$w = A v = A_{ij} v_j \qquad w_i = A_{ij} v_j$$

37.) $U = e^{i\hat{H} t} = ..$

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

$$U = 1 + iHt + \frac{1}{2!}(iHt)^2 + \dots$$

$$U^+ U = \left(1 - i H^+ t - \frac{1}{2!} H^{+2} t^2 + \dots\right)\left(1 + iHt + \frac{1}{2!}(iHt)^2 + \dots\right)$$

$$= 1 + i(H - H^+)t + t^2 \underbrace{\left\{-\frac{1}{2} H^{+2} - \frac{1}{2} H^2 + H^+ H\right\}}_{0} + \dots$$

$= 1$

38.) Dinamika dvonivojskega sistema

$$H = -\vec{\mu}\vec{B} =$$

$$= +g_s\mu_B \vec{S}\vec{B} = g_s\mu_B S^z B^z$$

$$= \frac{g_s\mu_B}{2}\begin{pmatrix} B_z & 0 \\ 0 & -B_z \end{pmatrix}$$

$$\mu = +g_s \frac{-e_0}{2m_e}\Gamma =$$

$$= -g_s \frac{e_0\hbar}{2m_e}\vec{S} =$$

$$= -g_s \underbrace{\mu_B}_{\text{Bohrov magneton}} \vec{S}$$

$$g_s \approx 2$$

Zeemanova energija $\hbar\omega_0 = B_z \mu_B g_s$

$$H = \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lastne energije $E_\uparrow = \frac{\hbar\omega_0}{2}$

$E_\downarrow = -\frac{\hbar\omega_0}{2}$

Začetni pogoj $e^{-i\frac{\phi}{2}}|\psi(0)\rangle = e^{-i\frac{\phi}{2}}\left(\cos(\tfrac{\theta}{2})\,|\uparrow\rangle + \sin(\tfrac{\theta}{2})\,e^{i\frac{\phi}{2}}\,|\downarrow\rangle\right)$

$$= \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}\,|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\frac{\phi}{2}}|\downarrow\rangle$$

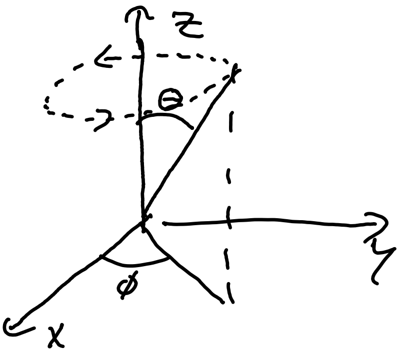
$$e^{iHt}\left(\alpha|\uparrow\rangle + \beta|\downarrow\rangle\right] = \alpha e^{-iE_1 t/\hbar}|\uparrow\rangle + \beta e^{-iE_2\frac{t}{\hbar}}|\downarrow\rangle$$

$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)e^{-i\frac{\phi}{2}}e^{\frac{-iE_\uparrow t}{\hbar}}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{+i\frac{\phi}{2}}e^{-i\frac{E_\downarrow t}{\hbar}}|\downarrow\rangle =$$

$$= \cos\left(\frac{\theta}{2}\right)e^{-i(\phi+\omega_0 t)/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i(\phi+\omega_0 t)/2}|\downarrow\rangle$$

Torej $\theta = \theta(0)$

$\phi(t) = \phi + \omega_0 t$



Pričakovane vrednosti spina

$$\langle S_z(t)\rangle = \frac{1}{2}\langle\psi(t)|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}|\psi(t)\rangle =$$

$$= \frac{1}{2}\left[\cos\frac{\theta}{2}e^{\frac{i\phi}{2}} \quad \sin\frac{\theta}{2}e^{-\frac{i\phi}{2}}\right]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\,e^{-i\phi(t)/2}\\ \sin\left(\frac{\theta}{2}\right)e^{+i\phi(t)/2}\end{bmatrix} :$$

$$\langle S_z(t)\rangle = \frac{1}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cos\theta$$

$$\langle S_x(t)\rangle = \frac{1}{2}\left[\cos\frac{\theta}{2}\, e^{i\phi(t)/2} \quad \sin\frac{\theta}{2}\, e^{-i\phi(t)/2}\right]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\, e^{-i\phi(t)/2}\\ \\ \sin\frac{\theta}{2}\, e^{i\phi(t)/2}\end{bmatrix}$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left[e^{i\phi(t)} + e^{-i\phi(t)}\right] = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$
$$2\cos\left[\phi(t)\right) =$$

$$\langle S_x\rangle = \frac{1}{2}\sin(\theta)\cos(\phi(t))$$

$$\langle S_y = \frac{1}{2}\sin(\theta)\sin(\phi(t))$$

Pomen - Jedrska magnetna
resonanca

- popravki od okolice
(kemijski
premik)

Vaje 8

38.) Dinamika dvonivojskega sistema

$$H = -\vec{\mu}\vec{B} =$$

$$= +g_s\mu_B\vec{S}\vec{B} = g_s\mu_B S^z B^z$$

$$\mu = +g_s\frac{-e_0}{2m_e}\Gamma =$$

$$= -g_s\frac{e_0\hbar}{2m_e}\vec{S} =$$

$$= -g_s\underset{\downarrow}{\mu_B}\vec{S}$$

Bohrov magneton

$g_s \approx 2$

$$= \frac{g_s\mu_B}{2}\begin{pmatrix} B_z & 0 \\ 0 & -B_z \end{pmatrix}$$

Zeemanova energija $\hbar\omega_0 = B_z\mu_B g_s$

$$H = \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lastne energije $E_\uparrow = \frac{\hbar\omega_0}{2}$

$E_\downarrow = -\frac{\hbar\omega_0}{2}$

Začetni pogoj $e^{-i\frac{\phi}{2}}|\psi(0)\rangle = e^{-i\frac{\phi}{2}}\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\frac{\phi}{2}}|\downarrow\rangle\right)$

$$= \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\frac{\phi}{2}}|\downarrow\rangle$$

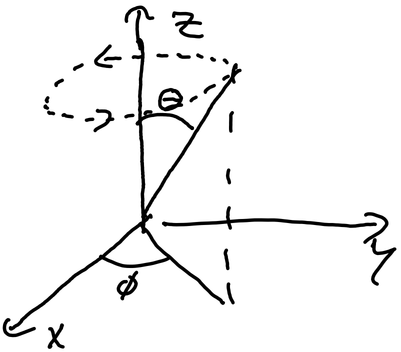
$$e^{iHt}\left(\alpha|\uparrow\rangle+\beta|\downarrow\rangle\right] = \alpha e^{-iE_1 t/\hbar}|\uparrow\rangle + \beta e^{-iE_2\frac{t}{\hbar}}|\downarrow\rangle$$

$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)e^{-i\frac{\phi}{2}}e^{\frac{-iE_\uparrow t}{\hbar}}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{+i\frac{\phi}{2}}e^{-i\frac{E_\downarrow t}{\hbar}}|\downarrow\rangle =$$

$$= \cos\left(\frac{\theta}{2}\right)e^{-i(\phi+\omega_0 t)/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i(\phi+\omega_0 t)/2}|\downarrow\rangle$$

Torej $\theta = \theta(0)$

$\phi(t) = \phi + \omega_0 t$



Pričakovane vrednosti spina

$$\langle S_z(t)\rangle = \frac{1}{2}\langle\psi(t)|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}|\psi(t)\rangle =$$

$$= \frac{1}{2}\left[\cos e \ \ \sin e\right]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\, e^{-i\phi(t)/2}\\ \sin\left(\frac{\theta}{2}\right)e^{+i\phi(t)/2}\end{pmatrix} :$$

$$\langle S_z(t)\rangle = \frac{1}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cos\theta$$

$$\langle S_x(t)\rangle = \frac{1}{2}\left[\cos\frac{\theta}{2}\, e^{i\phi(t)/2} \quad \sin\frac{\theta}{2}\, e^{-i\phi(t)/2}\right]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2}\, e^{-i\phi(t)/2} \\ \sin\frac{\theta}{2}\, e^{i\phi(t)/2} \end{bmatrix}$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left[e^{i\phi(t)} + e^{-i\phi(t)}\right] = \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) 2\cos\left[\phi(t)\right) =$$

$$\langle S_x\rangle = \frac{1}{2}\sin(\theta)\cos(\phi(t))$$

$$\langle S_y = \frac{1}{2}\sin(\theta)\sin(\phi(t))$$

Pomen - jedrska magnetna resonanca

- popravki od okolice (kemijski premik)

Prehod navezja

Bellovo stanje ni separabilno

39.) $\psi = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right) =$

$$\left(\alpha |0\rangle + \beta |1\rangle \right) \otimes \left(\gamma |0\rangle + \delta |1\rangle \right) =$$

$$= \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle$$

$\alpha\gamma = \frac{1}{\sqrt{2}}$

$\alpha\delta = 0 \quad \Rightarrow \quad \alpha = 0$ ali $\delta = 0$ kontradikcija

$\beta\gamma = 0 \qquad \beta = 0$ ali $\gamma = 0$

$\beta\delta = \frac{1}{\sqrt{2}}$

40.) a) $|0\rangle, |1\rangle \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) 2-kubitna stanja

Dirac $|0\rangle \otimes |0\rangle = |00\rangle \qquad |1\rangle \otimes |0\rangle = |10\rangle$

$|0\rangle \otimes |1\rangle = |01\rangle \qquad |1\rangle \otimes |1\rangle = |11\rangle$

Bazni $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \ldots$

c) Delovanje NOT na kubitna vrata

$U_{not} |0\rangle = |1\rangle \qquad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

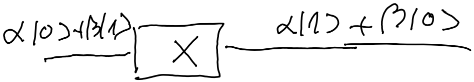
$U_{not} |1\rangle = |0\rangle \qquad U$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{matrix} a=0 \\ c=1 \end{matrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} b=1 \\ d=0 \end{matrix}$$

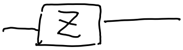
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X = \sigma_X$$



kvantna NOT vrata

d) $\sigma_z = Z$



$$\sigma_z(\alpha|0\rangle+\beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

phase-flip

e) $\sigma_y = Y$

$$\sigma_y(\alpha|0\rangle+\beta|1\rangle) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = -i\beta|1\rangle + i\alpha|0\rangle$$

f) Hadamarjev operator H
("adamar")

$$H|0\rangle = (|0\rangle+|1\rangle)/\sqrt{2}$$

$$H|1\rangle = (|0\rangle-|1\rangle)/\sqrt{2}$$

$H_{ij} = \langle i | H | j \rangle$

$\langle 0 | H | 0 \rangle = \frac{1}{\sqrt{2}}$

$\langle 1 | H | 0 \rangle = 1/\sqrt{2}$

$\langle 0 | H | 1 \rangle = \frac{1}{\sqrt{2}}$

$\langle 1 | H | 1 \rangle = -\frac{1}{\sqrt{2}}$

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$H^{\dagger} = H$

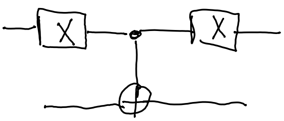
$H^2 = \mathbb{1}$

41.) $$H X H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

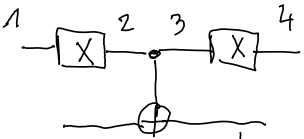
$$H Y H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} =$$
$$= \frac{i}{2} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -Y$$

$$H Z H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

42.)



12.)



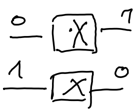
zgornja spodnja žička

$\downarrow \quad \downarrow$

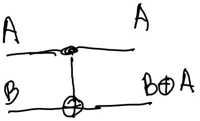
$|A \quad B\rangle$

Direktna vsota

NOT



CNOT



$B \oplus A = B + A$ mod 2

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle$

$\uparrow$ kontrolni ciljni

CNOT

$$\left(\begin{array}{cc|c} 1 & \emptyset & \emptyset \\ \emptyset & 1 & \\ \hline \emptyset & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right) = $$

1) $\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$] NOT

2.) $\alpha|10\rangle + \beta|1\cancel{0}1\rangle + \gamma|00\rangle + \delta|01\rangle$] CNOT

3) $\alpha|11\rangle + \beta|10\rangle + \gamma|00\rangle + \delta|01\rangle$] NOT

4) $\alpha|01\rangle + \beta|00\rangle + \gamma|10\rangle + \delta|11\rangle$

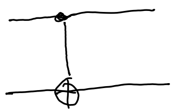
In	out
00	01
01	00
10	10
11	11

$\langle 01| M |00\rangle = 1$

$\langle 00| M |0\cancel{0}1\rangle = 1$

$$M = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \emptyset \\ \emptyset & \begin{matrix} 1 & \\ & 1 \end{matrix} \end{pmatrix}$$

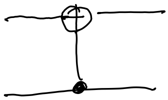
CNOT



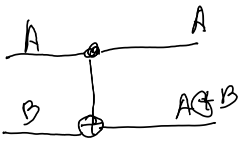
$$\text{CNOT} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

Naša matrica

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



43)



$\underline{A \oplus B = A + B \mod 2}$

$|A,B\rangle = |A, A\oplus B\rangle$

$|0\,0\rangle \to |0\,0\rangle$

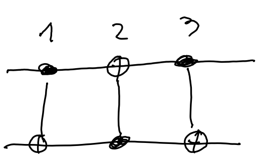
$|0\,1\rangle \to |0,1\rangle$

$|1\,0\rangle \to |1,1\rangle$

$|1\,1\rangle \to |1,0\rangle$

$$M = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$ – CNOT vrata

44.)



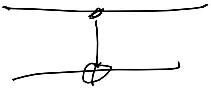
$|a,b\rangle \xrightarrow{1} |a, a\oplus b\rangle \xrightarrow{2}$

$\xrightarrow{2} |a\oplus(a\oplus b), a\oplus b\rangle = |b, a\oplus b\rangle$

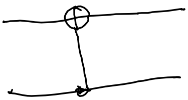
$a\oplus b\oplus b \equiv a \pmod 2$ $\xrightarrow{3} |b, a\oplus b\oplus b\rangle = |b,a\rangle$

SWAP : vrata

Alternativa z matrikami

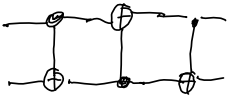


$$M = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & & \\ 0 & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$



a b	$a \oplus b$	b
0 0	0	0
0 1	1	1
1 0	1	0
1 1	0	1

$$\overline{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



$$M \quad \overline{M} \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$= M . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

9. vaje

→ $|\psi_1\rangle = |10\rangle$

→ $|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Enodelčna stanja

$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

$\boxed{45.}$

Dvodelčna stanja

$|\psi_1\rangle = |10\rangle$

$I \otimes I\, |\psi_1\rangle = |\psi_1\rangle$

$X \otimes I\, |10\rangle = |0\rangle \otimes |0\rangle = |00\rangle$

$X \otimes Z\, |10\rangle = |0\rangle \otimes |0\rangle = |00\rangle$

$Z \otimes X\, |10\rangle = (-1)|1\rangle \otimes |1\rangle = -|11\rangle$

$H \otimes I\, |10\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right)$

$H \otimes H\, |10\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left(|0\rangle - |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) = \frac{1}{2}\big(|00\rangle + |01\rangle$
$- |10\rangle - |11\rangle\big)$

$$\Rightarrow |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$X \otimes I \; |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right)$$

$$X \otimes Z \; |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$$

$$Z \otimes X \; |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$\rightarrow \quad \underline{\underline{H}} \otimes I \; |\psi_2\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left(|0\rangle + |1\rangle\right) \otimes |0\rangle + \left(\frac{1}{\sqrt{2}}\right)^2 \left(|0\rangle - |1\rangle\right) \otimes |1\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left(|00\rangle + |10\rangle + |01\rangle - |11\rangle\right)$$

$$\rightarrow \quad H \otimes H \; |\psi_2\rangle = \left(\frac{1}{\sqrt{2}}\right) \left[H|0\rangle \otimes H|0\rangle + H|1\rangle \otimes H|1\rangle \right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left[\underline{\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)} + \underline{\left(|0\rangle - |1\rangle\right) \otimes \left(|0\rangle - |1\rangle\right)} \right] =$$

$$= \frac{1}{2\sqrt{2}} \left[\underline{|00\rangle + \cancel{|01\rangle} + \cancel{|10\rangle} + |11\rangle} + \underline{|00\rangle - \cancel{|01\rangle} - \cancel{|10\rangle} + |11\rangle} \right]$$

$$= \frac{1}{\sqrt{2}} \left[|00\rangle + |11\rangle \right] = |\psi_2\rangle \quad \rightarrow \quad \lambda = 1$$

46) Deutsch algoritem (2 kubita) (Deutsch – Jozsa algoritem)

Orakelj: $f(000\;100\;11) = \begin{cases} 1 \\ 0 \end{cases}$

a) konstantna $f(0\;01\;01\;01) = 1$ $\quad f(001001) = 1$ $\quad f(*) = 1$

b) Polovica/polovica $f(001\;01\;01) = 1$ $\quad f(001001) = 0$ $\quad f(x) = \begin{cases} 0 \\ 1 \end{cases}$

Naloga: Določi, če je funkcija konstantna ali ne.

klasično : 2^n ; $2^{n-1}+1$

a) $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

b) $H \otimes H \ |00\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) =$

$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$H \otimes H \ |11\rangle = \frac{1}{2}\left(|0\rangle - |1\rangle\right) \otimes \left(|0\rangle - |1\rangle\right) =$

$= \frac{1}{2}\left(|00\rangle - |10\rangle - |01\rangle + |11\rangle\right)$

$H \otimes H \ |01\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$

$H \times H \ |10\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)$

$\rightarrow$ c) $\boxed{\underline{U_f} \ |a b\rangle = |a, b + f(a)\rangle \mod (2)}$ To predpostavimo

$$\underline{U_f}\ \underbrace{\left(H \otimes H \ |01\rangle\right)}\overset{?}{=} \frac{1}{2}\left(|00\rangle + (-1)^{f(0)+f(1)}|10\rangle - |01\rangle - (-1)^{f(0)+f(1)}|11\rangle\right)$$

$a \oplus b \equiv a + b \pmod 2$

1.) Delovanje U_f na vsa bazna stanja

2.) $U_F\,(H \otimes H)\,|01\rangle$

1.) $U_f\ |00\rangle = |0, 0 \oplus f(0)\rangle \doteq |0, f(0)\rangle$

2.) $U_f\ |01\rangle = |0, 1 \oplus f(0)\rangle$

$$U_f\,|10\rangle = |1, f(1)\rangle$$

$$U_f\,|11\rangle = |1, 1\oplus f(1)\rangle$$

$$f(*) = \begin{cases} 0 \\ 1 \end{cases}$$

$$U_f\left(|00\rangle - |01\rangle\right) = |0, f(0)\rangle - |0, 1\oplus f(0)\rangle =$$

$$= \begin{cases} |00\rangle - |01\rangle \\ |01\rangle - |00\rangle \end{cases} = (-1)^{f(0)}\,|0\rangle \otimes \left(|0\rangle - |1\rangle\right)$$

$$\rightarrow U_f\left(|10\rangle - |11\rangle\right) = |1, f(1)\rangle - |1, 1\oplus f(1)\rangle =$$

$$= \begin{cases} |10\rangle - |11\rangle \\ |11\rangle - |10\rangle \end{cases} = (-1)^{f(1)}\,|1\rangle \otimes \left(|0\rangle - |1\rangle\right)$$

1.) $U_F\,(H\otimes H)\,|01\rangle = U_F\,\frac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right) =$

$$= \frac{1}{2}\left[(-1)^{f(0)}\left(|00\rangle - |01\rangle\right) + (-1)^{f(1)}\left(|10\rangle - |11\rangle\right)\right] =$$

$$= \frac{1}{2}\underbrace{(-1)^{f(0)}}_{\text{globalna faza}}\left[|00\rangle - |01\rangle + (-1)^{f(1)\ominus f(0)}\left(|10\rangle - |11\rangle\right)\right] =$$

$$= (-1)^{f(0)}\,\frac{1}{2}\left[\underbrace{\left\{|0\rangle + (-1)^{f(0)\ominus f(1)}|1\rangle\right\}}_{\text{kanał 1}} \otimes \underbrace{\left(|0\rangle - |1\rangle\right)}_{\text{kanal 2}}\right]$$

$$f(0) \ominus f(1) = f(0) \oplus f(1)$$

d) $H \left(|0\rangle + (-1)^{f(0)\oplus f(1)} |1\rangle \right) =$

$$\frac{1}{\sqrt{2}} \left[|0\rangle + |1\rangle + (-1)^{f(0)\oplus f(1)} \left(|0\rangle - |1\rangle \right) \right] =$$

$$= \frac{1}{\sqrt{2}} \left[\underline{\left(1 + (-1)^{f(0)\oplus f(1)}\right)} |0\rangle + \underline{\left(1 - (-1)^{f(0)\oplus f(1)}\right)} |1\rangle \right]$$

$$f(0) \oplus f(1) = \begin{cases} 0 \\ 1 \end{cases}$$

$$H \left(|0\rangle - |1\rangle \right) = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle - |0\rangle - |1\rangle \right) = 0$$

$$\underline{\left(1 + (-1)^{f(0)\oplus f(1)}\right)} |0\rangle + \underline{\left(1 - (-1)^{f(0)\oplus f(1)}\right)} |1\rangle$$

Konstruktivno/destruktivna interferenca

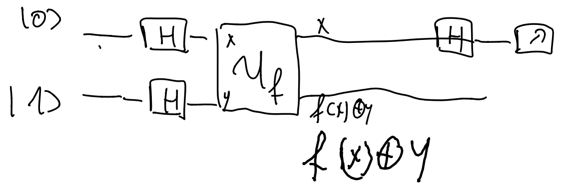
Če izmerim $|0\rangle$ $f(0) \oplus f(1) = 0$

$|1\rangle$ $f(0) \oplus f(1) = 1$

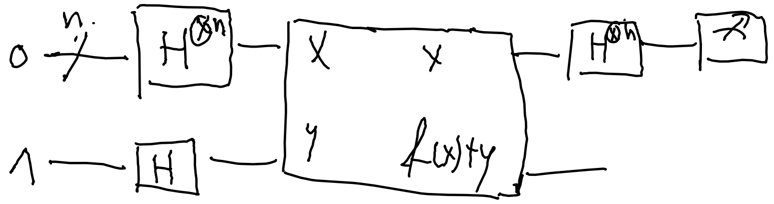
Ekvivalenca:

$f(0) = f(1)$ $f(0) \oplus f(1) \equiv 2f(0) \equiv 0 \mod (2)$

Implementacija:



Deutsch - Joscha algoritem (n bits



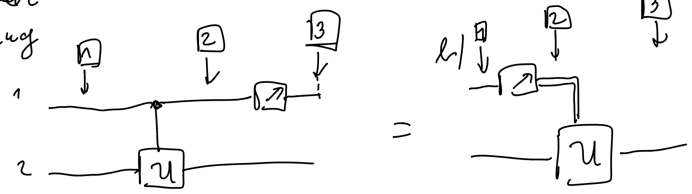
4.35 Meritev komutira z kontrola

Nielsen chang

U gate z lastnimi vrednostmi ±1

- Dvojna linija je klasični bit

a)



a) $\boxed{1}$ $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$\boxed{2}$ $\alpha_{00}|0\,U(0)\rangle + \alpha_{01}|0\,U(1)\rangle + \alpha_{10}|1\,U(0)| + \alpha_{11}|1\,U(1)\rangle$

Meritev

$\boxed{3}$ $|0\rangle$: $\alpha_{00}|U(0)\rangle + \alpha_{01}|U(1)\rangle$ – ✓

$|1\rangle$: $\alpha_{10}|U(0)\rangle + \alpha_{11}|U(1)\rangle$ – ✓

b/ $\boxed{1}$ $\underline{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle}$

$\boxed{2}$ $|0\rangle$: $\alpha_{00}|0\rangle + \alpha_{01}|1\rangle$

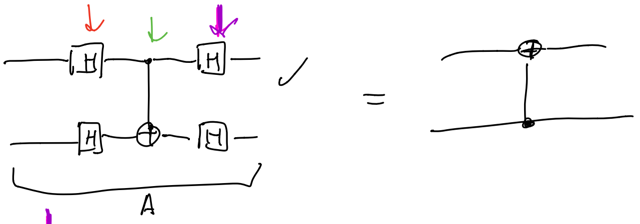
$|1\rangle$: $\alpha_{10}|0\rangle + \alpha_{11}|1\rangle$

$\boxed{3}$ $|0\rangle$: $\alpha_{00}|U(0)\rangle + \alpha_{01}|U(1)\rangle$ – ✓

$|1\rangle$: $\alpha_{10}|U(0)\rangle + \alpha_{11}|U(1)\rangle$ – ✓

$\boxed{4.20}$

N.C



$A = (H \otimes H) \cdot C_{Not} \cdot \underline{(H \otimes H)}$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$C_{Not} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & X \end{bmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & X \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \begin{bmatrix} H & H \\ XH & -XH \end{bmatrix} =$$

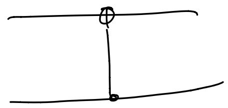
$$= \frac{1}{2} \begin{bmatrix} H^2 + HXH & H^2 - HXH \\ H^2 - HXH & H^2 + HXH \end{bmatrix}$$

$$HXH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = Z$$

$H^2 = \mathbb{1}$

$$A = \frac{1}{2} \begin{bmatrix} \mathbb{1} + Z & \mathbb{1} - Z \\ \mathbb{1} - Z & \mathbb{1} + Z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}}$$

$b|$



$CNOT\ |a, b\rangle \rightarrow |a, a \oplus b\rangle$

$CNOT_2\ |a, b\rangle \rightarrow |a \oplus b, b\rangle$

$|a\oplus$

$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |11\rangle$

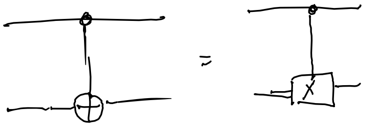
$|10\rangle \rightarrow |10\rangle$

$|11\rangle \rightarrow |01\rangle$

$$\underline{\underline{\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \left(\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}\right) \end{array}}}$$

Kontrolna Paulijeva vrata

CNOT = kontrolna X vrata



$|00\rangle \to |00\rangle$

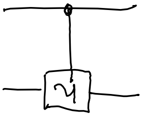
$|01\rangle \to |01\rangle$

$|10\rangle \to |11\rangle$

$|11\rangle \to |10\rangle$

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ 0 & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

Kontrolna U vrata (splošna unitarna operacija U)



$|00\rangle \to |00\rangle$

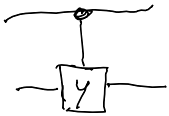
$|01\rangle \to |01\rangle$

$|10\rangle \to |1\rangle \otimes (u_{00}|0\rangle + u_{10}|1\rangle)$

$|11\rangle = |1\rangle \otimes (u_{01}|0\rangle + u_{11}|1\rangle)$

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & u_{00} & u_{10} \\ 0 & & u_{01} & u_{11} \end{pmatrix}$$

Kontrolna Y/Z vrata



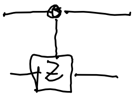
$|00\rangle \to |00\rangle$

$|01\rangle \to |01\rangle$

$|10\rangle \to i|11\rangle$

$|11\rangle \to -i|10\rangle$

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & 0 & -i \\ 0 & & i & 0 \end{pmatrix}$$



$|00\rangle \to |00\rangle$

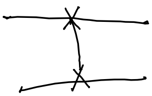
$|01 \to |01\rangle$

$|10\rangle \to |10\rangle$

$|11\rangle \to -|11\rangle$

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & 1 & 0 \\ 0 & & 0 & -1 \end{pmatrix}$$

SWAP gates



$|00\rangle \to |00\rangle$

$|01\rangle \to |10\rangle$

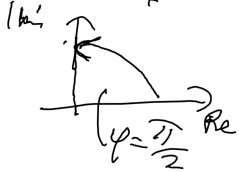
$|10\rangle \to |01\rangle$

$|11\rangle \to |11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

T-gates $\left(\frac{\pi}{8}\right)$ $-\boxed{T}-$ $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$

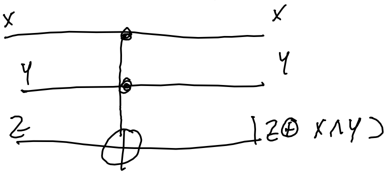
Phase gates (S, P) $-\boxed{S}-$ $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$



Toffoli gates (

$|a,b,c\rangle \rightarrow |a, b, c \oplus (a \wedge b)\rangle$

Kvantna vrata morajo biti reverzibilna



NOT ~~jeso~~ reverzibilna vrata

AND niso reverzibilna vrata

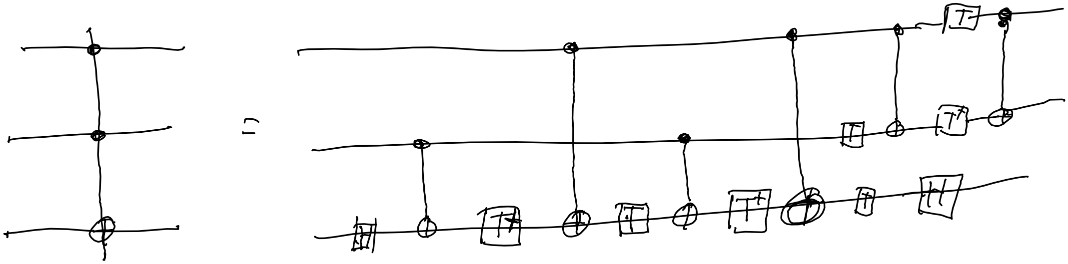
In	OUT
00	
01	→ 0
10	

INPUT			OUTPUT		
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

$$\begin{bmatrix} 1 &&&&&&& \\ & 1 &&&& 0 && \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ & 0 &&&&& 0 & 1 \\ &&&&&& 1 & 0 \end{bmatrix}$$

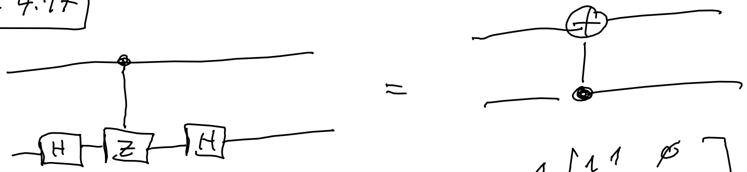
Simulacij AND, če z=0.

Realizacije (ena možna)



Naloga: Pokaži ekvivalenco

N.C 4.17



Matrično: $(1\!\!1 \otimes H)\cdot\left(|0\rangle\langle 0|1\!\!1 + |1\rangle\langle 1|Z\right)\cdot\left(1\!\!1 \otimes H\right) = \frac{1}{2}\begin{bmatrix} 1 & 1 & & 0 \\ 1 & -1 & & \\ & & 1 & 1 \\ 0 & & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & 1 & 0 \\ 0 & & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & & 0 \\ 1 & -1 & & \\ & & 1 & 1 \\ 0 & & 1 & -1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}\begin{bmatrix} H & 0 \\ 0 & \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & & 0 \\ 0 & 2 & & \\ & & 0 & 2 \\ 0 & & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & 0 & 1 \\ 0 & & 1 & 0 \end{bmatrix} = CNOT$

Stanja:

$|00\rangle \xrightarrow{H_2^{*}} \frac{1}{\sqrt{2}}|0\rangle\otimes(|0\rangle+|1\rangle) \xrightarrow{CZ} \frac{1}{\sqrt{2}}|0\rangle\otimes(|0\rangle+|1\rangle) \xrightarrow{H_2} \frac{1}{2}|0\rangle\otimes(|0\rangle+|1\rangle+|0\rangle-|1\rangle)$

$= |00\rangle$

$|01\rangle \xrightarrow{H_2} \frac{1}{\sqrt{2}}(|0\rangle\otimes(|0\rangle-|1\rangle) \xrightarrow{CZ} \frac{1}{\sqrt{2}}|0\rangle\otimes(|0\rangle-|1\rangle) \rightarrow \frac{1}{2}|0\rangle\otimes(|0\rangle+|1\rangle - |0\rangle+|1\rangle) =$

$\rightarrow |01\rangle$

$|10\rangle \rightarrow \frac{1}{\sqrt{2}}|1\rangle\otimes(|0\rangle+|1\rangle) \rightarrow \frac{1}{\sqrt{2}}|1\rangle\otimes(|0\rangle-|1\rangle) \rightarrow |11\rangle$

$$|11\rangle \to \frac{1}{\sqrt{2}} 1 \otimes . (|0\rangle - |1\rangle) \to \frac{1}{\sqrt{2}} |1\rangle \otimes (|0\rangle - |1\rangle) =$$

$$\to |10\rangle$$

Kvantni algoritem iskanja ("How the quantum search algorithm works")
(Grover algoritem 1996)

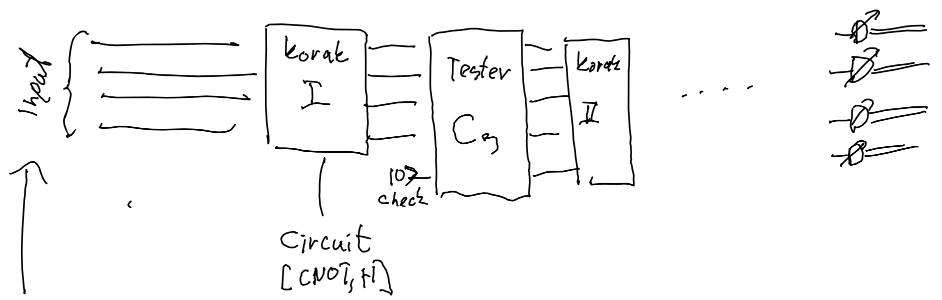
. Nestrukturirano iskanje inputa v black box, ki zadošča določenem odgovoru z uporabo $O(\sqrt{N})$ evalvacij funkcije z visoko verjetnostjo.

Klasična kompleksnost: $O(N)$ - vsaj $\frac{1}{2}$ je potrebno pregledati.

Uporaba: npr. brute force search preko 128-bit kriptografski ključ $\sim 2^{64}$ iteracij

Skica

Meritev



$|\psi\rangle = \sum_x \alpha_x |x\rangle$

$|x\rangle = |x_1 \dots x_n\rangle$
Bit string

$\boxed{\text{ORACLE = BLACK BOX}}$

C_s: Tester : $|x, 0\rangle \to \begin{cases} |x\,1\rangle \text{ ; če } x \text{ zadošča kriteriju} \\ |x\,0\rangle \text{ ; če ~~x~~ ne zadošča} \end{cases} = |x\, s(x)\rangle$

$s(x)$ .. search funct

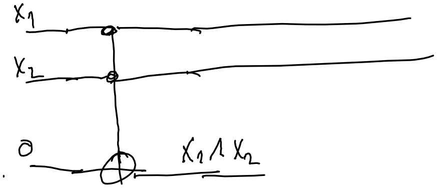
→ $S(x)$ bi implementirali kot v klasična vezja
(Toffoli + NOT vrata)

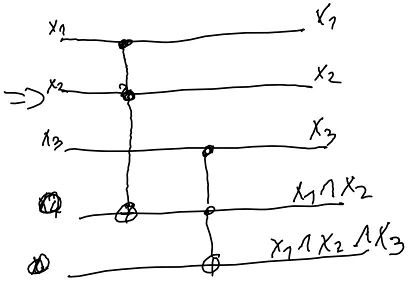
→ Prednost: algoritem deluje za katerikoli C_s (black box)

→ C_s mora samo prepoznati rešitev ! $\boxed{\text{oracle}}$

Predpostavka: obstaja točno 1 rešitev s.

Primer: $s(x) = x_1 \wedge x_2 \wedge x_2$





$|x_1 x_2 x_3 00\rangle \to |x_1, x_2, x_3, x_1 \wedge x_2, x_1 \wedge x_2 \wedge x_3\rangle$

Problem: vpleteni so dodatni kubiti:

$|x\,00\rangle \to |x\ s(x)\ w(x)\rangle$

⋮

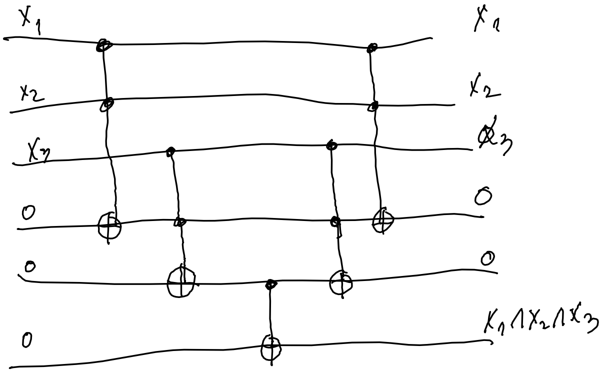
Kako se temu izognemo? Uncomputation:

Algoritem: 1.) $|x,0,0\rangle \to |x\ s(x)\ w(x)\rangle$ z standardnimi AND ⟺ Toffoli
NOT → X

2.) Dodaj ekstra kubit v stanje $|0\rangle$ in izvedi CNOT
z $s(x)$ ⇒ $|x\ s(x)\ w(x)\ s(x)\rangle$

3.) Apliciraj vrata iz koraka 1 v obratnem redu ⇒ $|x\,0\,0\ s(x)\rangle$

Primer $P(x) = x_1 \wedge x_2 \wedge x_3$



Splošen odgovor

$$|X\rangle \otimes |z\rangle \rightarrow |X\rangle \otimes |z \oplus \delta(x)\rangle$$

Primer

a) $|x_1 x_2\rangle |0\rangle \rightarrow |x_1\ x_2\rangle |x_1 \vee x_2\rangle$

b) $\delta(x_1 x_2 x_3) = x_1 \vee x_2 \vee x_3$

a) $|a, b, c\rangle \rightarrow |a, b, c \oplus (a \vee b)\rangle$

a	b	c	$c \oplus (a \vee b)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

za $\underline{c=0}$

$|000\rangle \rightarrow |000\rangle$

$|001\rangle \rightarrow |001\rangle$

$|010\rangle \rightarrow |011\rangle$

$|011\rangle \rightarrow |010\rangle$

$|100\rangle \rightarrow |101\rangle$

$|101\rangle \rightarrow |100\rangle$

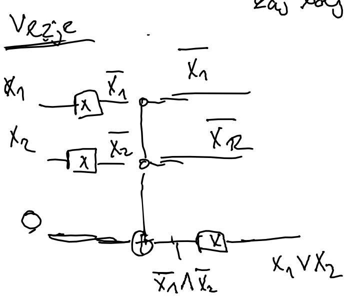
$|110\rangle \rightarrow |111\rangle$

$|111\rangle \rightarrow |110\rangle$

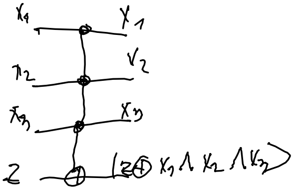
QOR
$$\begin{pmatrix} \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 & 0 & 0 \\ 0 & \begin{matrix}0 & 1\\ 1 & 0\end{matrix} & 0 & 0 \\ 0 & 0 & \begin{matrix}0 & 1\\ 1 & 0\end{matrix} & 0 \\ 0 & 0 & 0 & \begin{matrix}0 & 1\\ 1 & 0\end{matrix} \end{pmatrix}$$

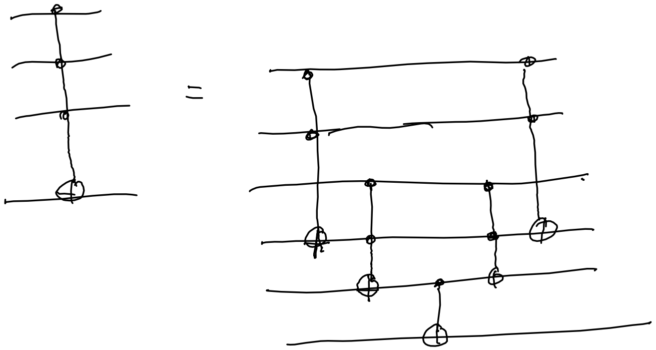
kaj bolj enostavnega?

Vezje

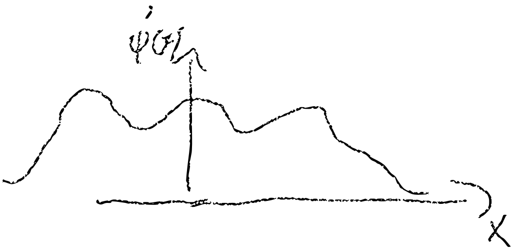


Generalizira Toffoli:





Vaje 10



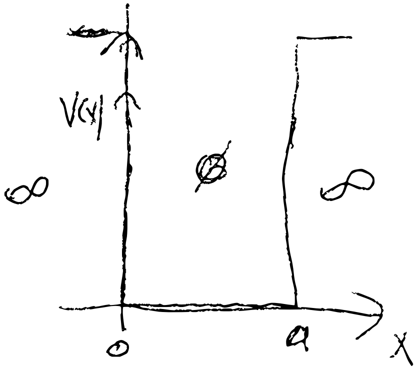
$$\hat{H}\Psi = E\Psi$$

↓

$$-\frac{\hbar^2 \partial^2}{2m\,\partial x^2}\psi(x) = E\psi(x)$$

$\boxed{47.}$

$\boxed{1D}$



$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$$

R.p.: $\underline{\underline{\psi(0) = \psi(a) = 0}}$

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$$\frac{\partial^2}{\partial x^2}\psi(x) = -Ak^2\sin(kx) - Bk^2\cos(kx)$$

$$+\frac{\hbar^2 k^2}{2m}\left[A\cancel{\sin(kx) + B}\cos(kx)\right] = E\left(A\cancel{\sin(kx)} + B\cos(kx)\right]$$

$$E = \frac{\hbar^2 k^2}{2m}$$

R.p.: $\psi(0) = B = 0$

$\psi(a) = 0 = A\sin(ka)$

$ka = \pi n$

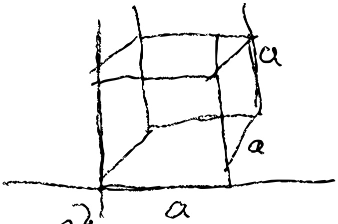
$k = \frac{\pi n}{a}$ $n = 1, 2, \ldots$

$$E = \underbrace{\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2}_{E_0} n^2 \quad ; \quad n = 1, 2, 3, \ldots$$

$$= E_0 n^2$$

$\boxed{3D}$

$$E = \underbrace{\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2}_{E_0}\left[n_x^2 + n_y^2 + n_z^2\right]$$



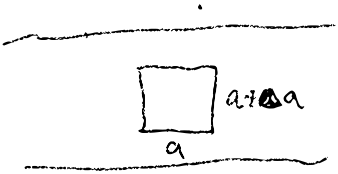
$$n_x, n_y, n_z \in \{1, 2, \ldots, \infty\}$$

Degeneracija = več stanj ima isto energijo

n_x	n_y	n_z	E/E_0	degeneracija
1	1	1	3	1
1 1 2	1 2 1	2 1 1	6	3
1 . .	2 . .	2 . .	9	3
1 .	1 .	3 .	11	3

2.) Imamo 2D potencialno jamo $\quad a,\ b = a+\Delta a$

$$\frac{\Delta a}{a} \ll 1.$$



a) kako se energije spremenijo za majhen $\frac{\Delta a}{a} \ll 1$

b) kakšna je razlika energij za obrnjen pravokotnik.

a)

$$E = \frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n_x}{a}\right)^2 + \left(\frac{n_y}{a+\Delta a}\right)^2\right] = \frac{\hbar^2\pi^2}{2ma^2}\left[n_x^2 + \left(\frac{n_y}{1+\frac{\Delta a}{a}}\right)^2\right] =$$

$$\frac{1}{\left(1+\frac{\Delta a}{a}\right)^2} = 1 - 2\frac{\Delta a}{a}$$

$$E = \frac{\hbar^2\pi^2}{2ma^2}\left[n_x^2 + n_y^2\left(1 - \frac{2\Delta a}{a}\right)\right] = \overbrace{\frac{\hbar^2\pi^2}{2ma^2}\left[n_x^2 + n_y^2\right]}^{\text{kvadrat s stranico } a}$$

$$- \frac{\hbar^2\pi^2}{2ma^2}\ \frac{2\Delta a}{a} n_y^2$$

b)

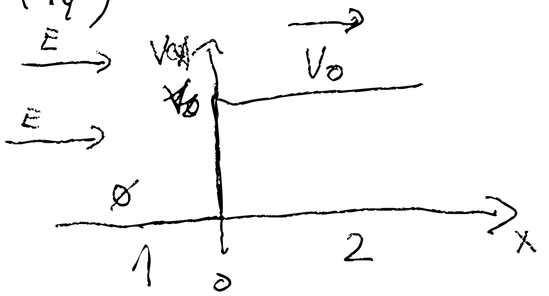
$$\Delta E = \left|E(n_x, n_y) - E(n_y, n_x)\right| =$$

$$= E_0\left[\cancel{n_x^2} + \cancel{n_y^2} - n_y^2\frac{2\Delta a}{a} - \left(\cancel{n_y^2} + \cancel{n_x^2} - 2\frac{\Delta a}{a} n_x^2\right)\right]$$

$$\approx E_0 \frac{2\Delta a}{a}\left(n_x^2 - n_y^2\right)$$

$$\frac{\Delta E}{E_{\Delta a>0}} = \frac{\cancel{E_0}\,(n_x^2-n_y^2)\,2\Delta a/a}{\cancel{E_0}\,(n_x^2+n_y^2)} = \frac{2\Delta a}{a}\,\frac{n_x^2-n_y^2}{n_x^2+n_y^2}$$

$\boxed{48}$



a) $\underline{E > V_0}$

$$\psi_1(x) = \underline{A_1}\, e^{ik_1 x} + \underline{B_1}\, e^{-ik_1 x}$$

$$\psi_1'(x) = A_1 ik_1 e^{ik_1 x} - ik B_1 e^{-ik_1 x}$$

$$\psi_2(x) = A_2\, e^{ik_2 x}$$

$$\psi_2'(x) = A_2 ik_2 e^{ik_2 x}$$

1.) $E = \frac{\hbar^2 k^2}{2m}$ $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

2.) $E = \frac{\hbar^2 k^2}{2m} + V_0$ $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ $p = \hbar k$

$\underline{R.p.}$: 1. $\boxed{\begin{aligned}\psi_1(0) &= \psi_2(0)\\ \psi_1'(0) &= \psi_2'(0)\end{aligned}}$! zvezna in zvezno odvedljiva

$\boxed{Rp1}$ $A_1 + B_1 = A_2$

$\boxed{Rp2}$ $ik_1(A_1 - B_1) = ik_2 A_2$

$\boxed{1}$ $A_1 + B_1 = A_2$

$\boxed{2}$ $k_1 A_1 - k_1 B_1 = k_2 A_2$

$=$

$\boxed{1} \cdot k_1 + \boxed{2}$: $2 k_1 A_1 = k_1 A_2 + k_2 A_2$ $\qquad \frac{A_2}{A_1} = \frac{2 k_1}{k_1 + k_2}$

$\boxed{1}\, k_1 - \boxed{2}$; $2 k_1 B_1 = A_2 (k_1 - k_2)$ $\qquad \frac{B_1}{A_2} = \frac{k_1 - k_2}{2 k_1}$

$$\frac{B_1}{A_1} = \frac{B_1}{A_2} \cdot \frac{A_2}{A_1} = \frac{(k_1 - k_2)}{\cancel{2k_1}} \cdot \frac{\cancel{2k_1}}{k_1 + k_2} =$$

$$= \frac{k_1 - k_2}{k_1 + k_2}$$

· Transmitivnost

$$T = \frac{j^2_{2D}}{j^2_{1D}} = \frac{k_2 |A_2|^2}{k_1 |A_1|^2}$$

Reflektivnost

$$R = \frac{J^2_{1L}}{J^2_{1D}} = \frac{k_1 |B_1|^2}{k_1 |A_1|^2}$$

Tak: $j = \mathrm{Im}\left[\psi^* \frac{\partial}{\partial x} \psi\right]$

$$\underline{J_{1D}} = \mathrm{Im}\left[A^* e^{-ik_1 x} \frac{\partial}{\partial x} A e^{ik_1 x}\right] =$$

$$= \mathrm{Im}\left[\underbrace{A^* A}_{|A|^2} \underbrace{e^{-ik_1 x} e^{ik_1 x}}_{1} i k_1\right] = \underline{k_1 |A|^2}$$

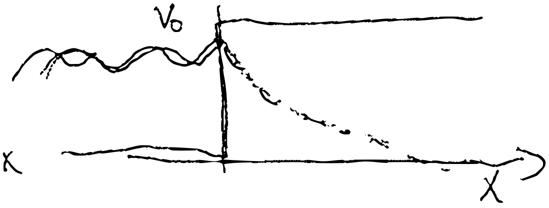
$$T = \frac{k_2 |A_2|^2}{k_1 |A_1|^2} = \frac{k_1}{k_1} \frac{4k_1^2}{(k_1+k_2)^2} = \underline{\frac{4k_1k_2}{(k_1+k_2)^2}}$$

$$R = \frac{k_1 |B_1|^2}{k_1 |A_1|^2} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

Pogoj: $R + T = 1$

$$\frac{\overset{= k_1^2+k_2^2-2k_1k_2}{(k_1-k_2)^2}}{(k_1+k_2)^2} + \frac{4k_1k_2}{(k_1+k_2)^2} = \frac{(k_1+k_2)^2}{(k_1+k_2)^2} = 1$$

b) $\underline{E < V_0}$



$$\psi_1(x) = A_1 e^{ik_1x} + B_1 e^{-ik_1x}$$

$$\psi_2(x) = A_2 e^{-\varkappa x}$$

$$\varkappa = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

R.p. $\psi_1(x \neq 0) = \psi_2(x=0)$

$\psi_1'(x=0) = \psi_2'(x=0)$

$$\boxed{\begin{aligned} A_1 + B_1 &= A_2 \\ ik_1(A_1 - B_1) &= -A\varkappa \end{aligned}}$$

$$\psi_2'(x) = A\, e^{-\varkappa x}(-\varkappa)$$

$\boxed{1}$ $A_1 + B_1 = A_2$

$\boxed{2}$ $A_1 ik_1 - B_1 ik_1 = -\varkappa A_2$

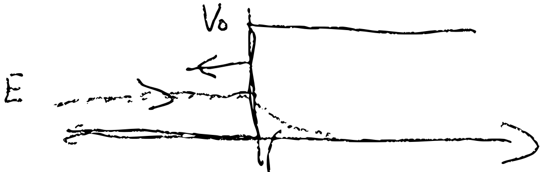
$\boxed{1}\varkappa + \boxed{2}$: $A_1(\varkappa + ik_1) + B_1(\varkappa - ik_1) = 0$

$$\frac{B_1}{A_1} = \frac{(\varkappa + ik_1)}{(\varkappa - ik_1)}$$

$$R = \frac{|B_1|^2 \cancel{k_1}}{|A_1|^2 \cancel{k_1}} = \frac{|B_1|^2}{|A_1|^2} = \frac{|\varkappa + ik_1|^2}{|\varkappa - ik_1|^2} = \frac{\varkappa^2 + k_1^2}{\varkappa^2 + k_1^2}$$

$$= 1$$

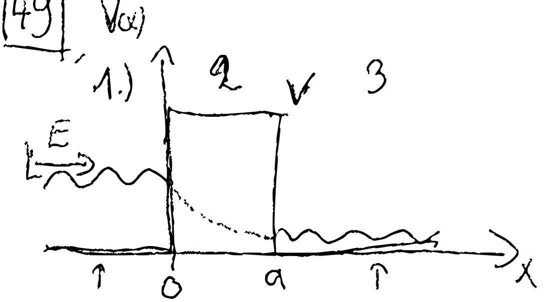
$\boxed{R = 1}$



eksponentno padajoča verjetnost v prepovedano območje

$$k = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

$\boxed{49}$



$\psi_1 = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$

$\psi_2 = A_2 e^{\varkappa_2 x} + B_2 e^{-\varkappa x}$

$\psi_3 = A_3 e^{ik_3(x-a)}$

$$T = \frac{|A_3|^2 k_3}{|A_1|^2 k_1} = ?$$

$\boxed{k_1 = k_3}$

R.p.:

$\boxed{1}\ \psi_1(x=0) = \psi_2(x=0)$; $A_1 + B_1 = A_2 + B_2$

$\boxed{2}\ \psi_1'(x=0) = \psi_2'(x=0)$; $ik_1(A_1 - B_1) = \varkappa_2 (A_2 - B_2)$

$\boxed{3}\ \psi_2(x=a) = \psi_3(x=a)$; $A_2 e^{\varkappa_2 a} + B_2 e^{-\varkappa_2 a} = A_3$

$\boxed{4}\ \psi_2'(x=a) = \psi_3'(x=a)$; $\varkappa_2 e^{\varkappa_2 a} A_2 - \varkappa_2 e^{-\varkappa_2 a} B_2 = ik_1 A_3$

$\boxed{1} \cdot ik_1 + \boxed{2}$ $\quad 2ik_1 A_1 = \underline{\underline{A_2}}(ik_1 + \varkappa_2) + B_2(ik_1 - \varkappa_2)$

$\rightarrow \boxed{3}\ \varkappa_2 + \boxed{4}$ $\quad 2A_2\varkappa_2 e^{\varkappa_2 a} = A_3(\varkappa_2 + ik_1)$

$\rightarrow \boxed{3}\ \varkappa_2 - \boxed{4}$ $\quad 2B_2\varkappa_2 e^{-\varkappa_2 a} = A_3(ik_1 - \varkappa_2)$

$$\underline{2ik_1} A_1 = (ik_1 + \varkappa_2)\frac{A_3(\varkappa_2 + ik_1)}{2\varkappa_2 e^{\varkappa_2 a}} + (ik_1 - \varkappa_2)\frac{A_3(ik_1 - \varkappa_2)}{2\varkappa_2 e^{-\varkappa_2 a}}$$

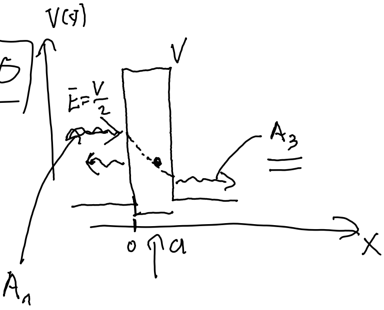
$$\frac{A_3}{A_1} = \frac{2ik_1 \cdot 2\varkappa_2}{(\varkappa_2 + ik_1)^2 e^{-\varkappa a} - (-\varkappa_2 + ik_1)^2 e^{\varkappa a}}$$

Široka oblast $\varkappa a \gg 1$

$$\frac{A_3}{A_1} = \frac{4ik_1\varkappa}{(-\varkappa_2 + ik_1)^2(-1)} e^{-\varkappa a} = C\, e^{-\varkappa a}$$

Vaje 11

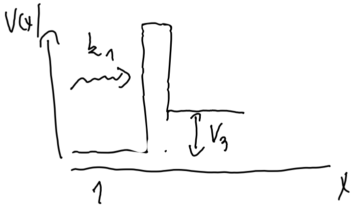
$\boxed{56}$



$$T = \frac{|A_3|^2 \cancel{k_1}}{|A_1|^2 \cancel{k_1}}$$

$$\rho = \frac{|A_3|^2}{|A_1|}$$

Transmitivna ≠ verjetnost



$$T = \frac{|A_3|^2 k_3}{|A_1|^2 k_1}$$

$$\rho = \frac{|A_3|^2}{|A_1|^2}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$k_3 = \sqrt{\frac{2m(E-V_3)}{\hbar^2}}$$

$\boxed{V=2E}$

$$\frac{A_3}{A_1} = \frac{4i k_1 \kappa_2}{(\kappa_2+ik_1)^2 \underset{\uparrow\, e^{-\kappa_2 a}}{} + (-\kappa_2+ik_1)^2 e^{\kappa_2 a}}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\kappa_2 = \sqrt{\frac{2m(V-E)}{\hbar^2}} = \sqrt{\frac{2mE}{\hbar^2}} = k_1$$

$$\frac{A_3}{A_1} = \frac{4i k_1^2}{k_1^2(1+i)^2 \underset{\uparrow\, e^{-\kappa_2 a}}{} + k_1^2(-1+i)\, e^{k_1 a}}$$

$$\rho = \frac{|A_3|^2}{|A_1|^2} = \frac{16\, \cancel{k_1^4}}{\cancel{k_1^4} \left|(1+i)^2 \underset{e^{-k_1 a}}{} + (-1+i)\, e^{k_1 a}\right|^2}$$

$$|(1+i)e^{-k_1a} - (1-i)e^{k_1a}|^2 = |(e^{-k_1a} - e^{k_1a}) + i(e^{-k_1a} + e^{k_1a})|^2$$

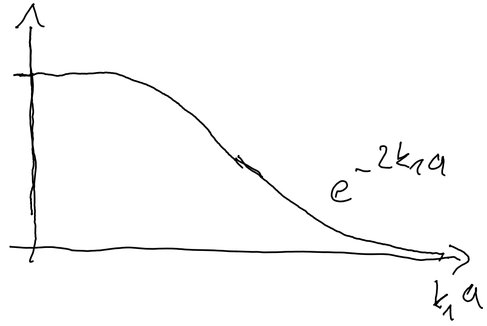
$$= (e^{-k_1a} - e^{k_1a})^2 + (e^{-k_1a} + e^{k_1a})^2 =$$

$$= e^{-2k_1a} + e^{2k_1a} \cancel{-2} + e^{-k_1a2} + e^{k_1a2} \cancel{+2} =$$

$$= 2(e^{2k_1a} + e^{-2k_1a})$$

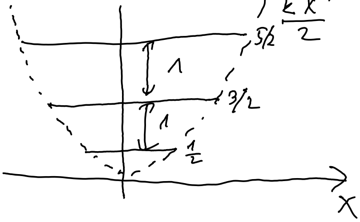
$$\cancel{A}\ \underline{\underline{p}} = \frac{|A_3|^2}{|A_1|^2} = \frac{\cancel{16}\ 8}{\cancel{2}(e^{2k_1a} + e^{-2k_1a})} = \underline{\frac{8}{e^{2k_1a} + e^{-2k_1a}}}$$

$$p(k_1a \to \infty) = \frac{8}{e^{2k_1a}} \sim 8e^{-2k_1a}$$



Poišči energije in ~~ge~~ degeneriranost v 3D harmonskem potencialu.

Klasično: $H = \frac{p^2}{2m} + \frac{kx^2}{2}$



kvantno: $\hat{H} = \frac{-\hbar^2\nabla^2}{2m} + \frac{1}{2}kx^2$

$\boxed{1D}$ $\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2$

$E_n = \hbar\omega_0(n + \frac{1}{2})$

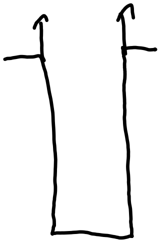
$\hbar\omega_0 = \hbar\sqrt{\frac{k}{m}}$

Ekvidistančne

$\boxed{3D}$ $H = -\frac{\hbar^2 \nabla^2}{2m} + V(\vec{r})$ $\rightarrow$ $\underline{E = \hbar\omega_0 (n_x + n_y + n_z + \frac{3}{2})}$

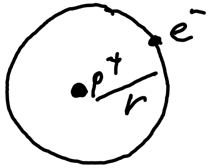
↓ ↓

$E/\hbar\omega_0$	(n_x, n_y, n_z)	Deg.
$3/2$	0, 0, 0	1
$5/2$	1 0 0 0 1 0 0 0 1	3
$7/2$	1 1 0 x3 2 0 0 x3	6
$9/2$	1 1 1 x1 2 1 0 x6 3 0 0 x3	10



$E = E_0 \cdot (n_x^2 + n_y^2 + n_z^2$

$\boxed{53}$



$$\rightarrow \underline{H = \frac{m v^2}{2} - \frac{e_0^2}{4\pi\varepsilon_0 r}}$$

Energija osnovnega stanja

$$\rightarrow \underline{\delta x \cdot \delta p \geq \frac{\hbar}{2}}$$

$$\delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Heissenbergov načelo nedoločenosti

$$\delta x \sim r$$

$$\delta p \geq \frac{\hbar}{2\delta x} = \frac{\hbar}{2\cdot r}$$

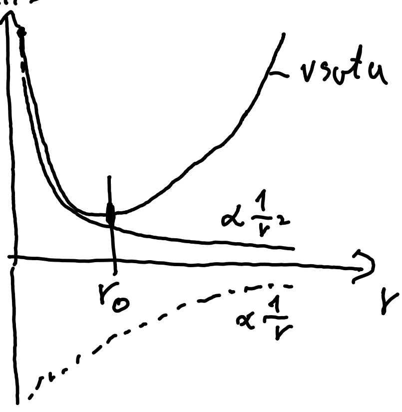
$$p = m v$$

$$\rightarrow \underline{\underline{\delta v}} \geq \frac{\hbar}{2m\underline{\underline{r}}}$$

$$\langle H\rangle = \left(\frac{\hbar}{2mr}\right)^2 \cdot \frac{m}{2} - \frac{e_0^2}{4\pi\varepsilon_0 r} = \frac{\hbar^2 \cancel{m}}{4m^2\cdot 2}\,\frac{1}{r^2} - \frac{e_0^2}{4\pi\varepsilon_0 \cancel{r}}$$

$$= \frac{\hbar^2}{8m}\,\frac{1}{r^2} - \frac{e_0^2}{4\pi\varepsilon_0}\,\frac{1}{r}$$

$$\frac{\partial\langle H\rangle}{\partial r} = 0 = \frac{\hbar^2}{2m}(-2)\frac{1}{r^3} - \frac{e_0^2}{4\pi\varepsilon_0}(-1)\frac{1}{r^2}$$



$$0 = \frac{\partial\langle H\rangle}{\partial r}\Big|_{r=r_0} = -\frac{2\hbar^2}{8m}\frac{1}{r^3} + \frac{e_0^2}{4\pi\varepsilon_0}\frac{1}{r^2} \qquad / \, r^3$$

Pogoj za minimum $$\frac{2\hbar^2}{8m} = \frac{e_0^2}{4\pi\varepsilon_0} r_0$$

$$\alpha = \frac{e_0^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$$

$$m_e c^2 = 0.5 \text{ MeV}$$

$$\hbar c = 0.197 \text{ eV}\mu\text{m}$$

$$r_0 = \frac{\hbar^2 4\pi\varepsilon_0}{4m e_0^2} =$$

$$r_0 = \frac{\hbar}{\underbrace{4m\hbar c}\;\frac{e_0^2}{4\pi\varepsilon_0\hbar c}} = \frac{1}{4}\,\frac{\frac{\hbar c}{mc^2}}{\alpha}$$

$$= \frac{1}{4}\,\frac{0.197 \text{ eV}\mu\text{m}/(0.5 \text{ MeV})}{1/137} = 1.3\cdot10^{-5}\mu\text{m} =$$

$$\underline{1.3\cdot10^{-11} \text{ m} \approx 0.1 \text{ nm}}$$

Bohrov radij

$$\underline{r_B = 5.2\cdot10^{-11} \text{ m}}$$

54.) $\Delta E = 0.306$ eV ; $n, n+1$; zanima nas n?

$$\hat H = -\frac{\hbar^2\nabla^2}{2m} - \frac{e_0^2}{4\pi\varepsilon_0 r} \rightarrow E_n = -\frac{E_0}{n^2} \qquad E_0 = 13.6 \text{ eV}$$

Rydberg konst.

$$\Delta E = 0.306\,eV = E_{n+1} - E_n = -E_0\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)$$

$$= -E_0 \,\frac{n^2 - (n+1)^2}{n^2\,(n+1)^2} = +E_0\,\frac{(+2n+1)}{n^2\,(n+1)^2} = \Delta E$$

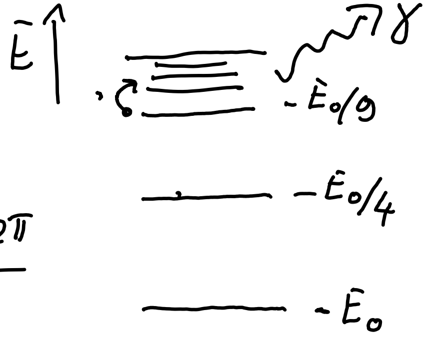
$$\frac{\Delta E}{E_0} = \frac{2n+1}{n^2(n+1)^2} = \frac{0.306\,eV}{13.6\,eV} \simeq 0.0225$$

n	$(2\cdot n+1)/(n^2(n+1)^2)$
1	3/4
2	0.138
3	0.0486
4	0.0225

$\Rightarrow \boxed{n=4}$

$\boxed{55.}$ $n=4 \rightarrow$

$\lambda = 485.3\ nm$



$$\parallel\ \Delta E = h\frac{c}{\lambda} = \frac{197\ eV\,\cancel{nm}\cdot 2\pi}{485.3\ \cancel{nm}}$$

$$= 0.406\ eV$$

$$\parallel\ -\Delta E = -E_0\left(\frac{1}{4^2} - \frac{1}{n^2}\right)$$

$$-\frac{\Delta E}{E_0} + \frac{1}{16} = \frac{1}{n^2}$$

$$n = \sqrt{\frac{1}{\frac{1}{16} - \frac{\Delta E}{E_0}}} = 6.05$$

$\boxed{n=6}$

$\boxed{56}$ Kakšen je najverjetnejši radij vodikovega atoma za lastna stanja (n=1, l=0)

(n=2, l=0)

(n=2, l=1)

$|n,l,m\rangle$

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e_0^2}{4\pi\varepsilon_0 r}$$

n ... radialno

l ... orbitalno

m ... projekcija vrtilne količine na z-os

$$\underline{|n,l,m\rangle} = \psi_{nlm}(r,\theta,\varphi)$$

$$= \underbrace{\sqrt{\left(\frac{2}{n a_0}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}}\; e^{-\rho/2} \rho^l L^{2l+1}_{n-l-1}(\rho)}_{R_{nl}(r)} \; Y_l^m(\theta,\varphi)$$

$\rho = \frac{r}{a_0}$ a_0 - Bohrov radij

L^{2l+1}_{n-l-1} - Laguerrov polinom

$Y_l^m(\theta,\varphi)$ - sferični harmoniki

Sferičnih koordinatah: $dV = r^2 dr\, d(\cos\vartheta)\, d\varphi$

$$\int_0^\infty R_{nl}^2 dV = \int R_{nl}^2(r) \cdot r^2 dr = 1$$

Normalizacija

$$\int_{-1}^{1} \int_0^{2\pi} |Y_{lm}|^2 d(\cos\vartheta) d\varphi = 1$$

$$R_{10}\left(\rho = \frac{r}{a_0}\right) = 2\, e^{-r/a_a} \qquad \boxed{n=1,\ \ell=0}$$

$$R_{20}(r) = \frac{1}{\sqrt{2}}\, e^{-\frac{r}{2a_a}} \left(1 - \frac{r}{2a_0}\right) \qquad \boxed{n=2,\ \ell=0}$$

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\, e^{-\frac{r}{2a_0}}\, \frac{r}{a_0} \qquad \boxed{n=2,\ \ell=1}$$

$$dP = |\psi|^2 \cdot dV = |\psi|^2\, r^2 dr \underbrace{\int_{-1}^{1} d(\cos\vartheta)}_{2} \underbrace{\int_{0}^{2\pi} d\varphi}_{2\pi}$$

$$= 4\pi\, |\psi|^2\, r^2 dr$$

$$\boxed{n=1,\ \ell=0} \qquad dP = 4\pi\, |R_{10}|^2 \cdot r^2 = 4\pi \cdot 4\, e^{-\frac{2r}{a_0}} \cdot r^2$$

$$\frac{dP}{dr} = 0 = 16\pi \left[2r\, e^{-\frac{2r}{a_0}} + r^2\, e^{-\frac{2r}{a_0}} \left(-\frac{2}{a_0}\right) \right] =$$

$$16\pi\, e^{-\frac{2r}{a_0}} \left[2r - 2\frac{r^2}{a_0} \right] = 0$$

$$36\pi\, e^{-\frac{2r}{a_0}}\, 2r \left[1 - \frac{r}{a_0} \right) = 0 \quad \Rightarrow \quad \boxed{r=0}\ X$$

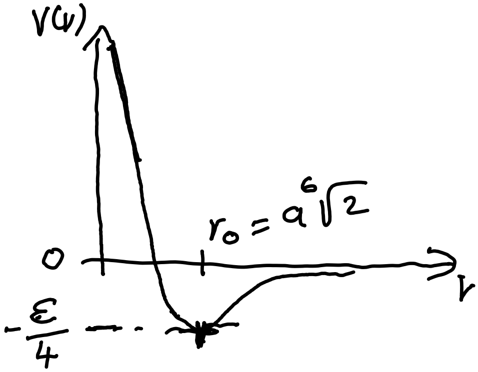
$$\underline{r = a_0}$$

Maximum pri a_0.

II kolokvij 21.5. ob 10.00

57. (8.5 iz Ambrožič)

$$V(r) = \varepsilon\left[\frac{a^{12}}{r^{12}} - \frac{a^6}{r^6}\right]$$



→ Minimum potenciala :

$$\frac{\partial V(r)}{\partial r} = 0 = \varepsilon \underbrace{\left[\frac{(-12)\,a^{12}}{r^{13}} - \frac{a^6 \cdot (-6)}{r^7}\right]}_{\substack{\| \\ 0}} \quad / \; r^{13}$$

$$-12\,\overset{a^6}{\cancel{a^{12}}} + 6\,\cancel{a^6}\, r^6 = 0$$

$$r^6 = \frac{12\,a^6}{6} = 2a^6 \qquad r = a\sqrt[6]{2}$$

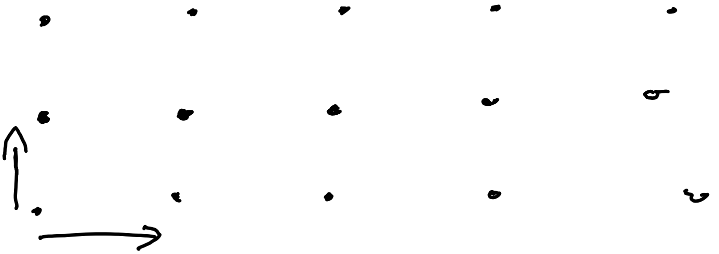
$$V(a\sqrt[6]{2}) = \varepsilon\left[\frac{a^{12}}{a^{12}\,2^2} - \frac{a^6}{a^6 \cdot 2}\right] = \varepsilon\left[\frac{1}{4} - \frac{1}{2}\right] =$$

$$= -\frac{\varepsilon}{4}$$

Asimptotsko obnašanje

$r \to \infty$: $\quad V(r) \approx -\varepsilon \frac{a^6}{r^6}$

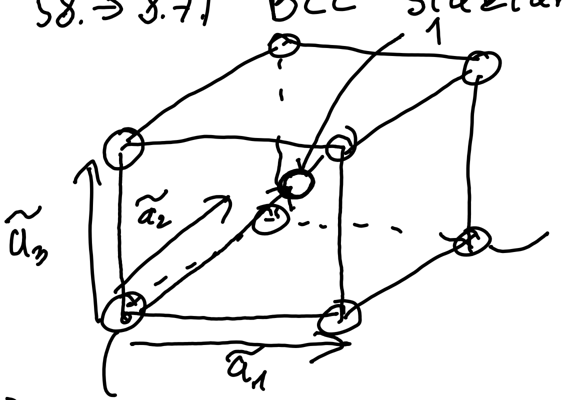
$r \to 0 \quad V(r) \approx +\varepsilon \frac{a^{12}}{r^{12}}$



$\{\vec{a}_1, \vec{a}_2\}$ $\vec{R} = n\,\vec{a}_1 + m\,\vec{a}_2$

$(n, m) \in \mathbb{Z}$

58. → 8.7.) BCC struktura



$8 \cdot \frac{1}{8} + 1 = 2$ atoma na celico

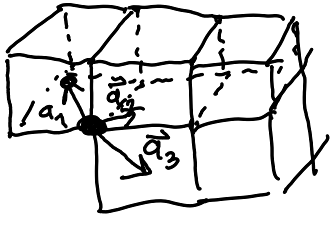
$\vec{r}_1 = (0,0,0)$; $\frac{a}{2}(1,1,1) = \vec{r}_2$

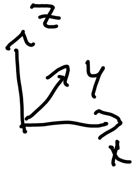
$\vec{a}_1 = \frac{a}{2}(-1,,1,1)$

$\vec{a}_2 = \frac{a}{2}(1,-1,1)$

$\vec{a}_3 = \frac{a}{2}(1,1,-1)$

Primitivna celica ≡ 1 atom na celico





Volumen celice

$$V = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)$$

$$\vec{a}_2 \times \vec{a}_3 = \left(\frac{a}{2}\right)^2 \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{a^2}{4}(0, +2, +2)$$

$$\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{8} \quad (-1, 1, 1)\,(0, 2, 2) = \frac{a^3 \cdot 4}{8} = \underline{\underline{\frac{a^3}{2}}}$$

1 atom
na celico
||
primitivna
celica

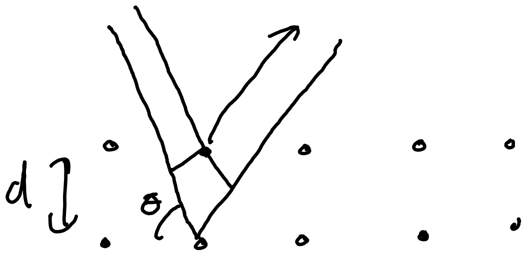
8.9 - Recipročna mreža

Bravaisova mreža $\quad r = n_1 \vec{a}_1 + n_2 \vec{a}_2 + n_3 \vec{a}_3$

$n_1, n_2, n_3 \in \mathbb{Z}$

Recipročno mrežo

$n\lambda = 2d\sin\theta$



Laue pogoj: $\boxed{\vec{k}_{ven} - \vec{k}_{notri} = \vec{G}}$ → konstruktivna interferenca

$$\underline{\vec{G} = h\vec{b}_1 + k\vec{b}_2 + l\vec{b}_3}$$

$$\| \; \vec{b}_1 = 2\pi \frac{\vec{a}_2 \times \vec{a}_3}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

$\vec{b}_i$ .. vektorji recipročne mreže

$$\| \; \vec{b}_2 = 2\pi \frac{\vec{a}_3 \times \vec{a}_1}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

(h,k,l) Millerjevi indeksi

$$\| \; \vec{b}_3 = 2\pi \frac{\vec{a}_1 \times \vec{a}_2}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

$(1,2,3)$

Naloga: Pokaži, da je recipročna mreža od recipročne mreže originalna mreža

$$\vec{C}_1 = 2\pi \frac{\vec{b}_2 \times \vec{b}_3}{\vec{b}_1 \cdot (\vec{b}_2 \times \vec{b}_3)} \overset{?}{=} \vec{a}_1$$

<u>Namig:</u>

$$(\vec{X} \times \vec{Y}) \times (\vec{Y} \times \vec{Z}) = [\vec{Y} \cdot (\vec{X} \times \vec{Z})] \, \vec{Y}$$

<u>Mešani produkt</u> $(\vec{Y}, \vec{X}, \vec{Z}) = \vec{Y} \cdot (\vec{X} \times \vec{Z}) = (\vec{Z}, \vec{Y}, \vec{X}) =$

$= (\vec{X}, \vec{Z}, \vec{Y})$

1.) $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$

1.) $$(\vec{b}_1, \vec{b}_2, \vec{b}_2) = \left[\frac{2\pi}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)}\right]^3 (\vec{a}_2 \times \vec{a}_3, \underbrace{\vec{a}_3 \times \vec{a}_1, \vec{a}_1 \times \vec{a}_2)}_{\times}$$

$$= \frac{(2\pi)^3}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)^3} (\vec{a}_2 \times \vec{a}_3) \cdot \underbrace{(\vec{a}_1, \vec{a}_2, \vec{a}_3)}_{\text{številk}} \cdot \vec{a}_1$$

$$= \frac{(2\pi)^3}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)^3} (\vec{a}_1, \vec{a}_2, \vec{a}_3) \cdot \underbrace{(\vec{a}_2 \times \vec{a}_3) \cdot \vec{a}_1}_{\substack{(\vec{a}_2, \vec{a}_3, \vec{a}_1) \\ \parallel \\ (\vec{a}_1, \vec{a}_2, \vec{a}_3)}} =$$

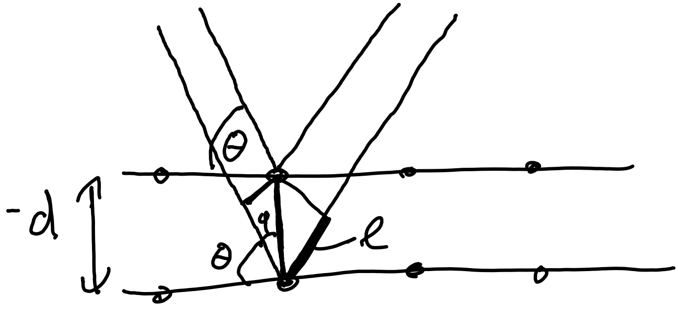
$$= \frac{(2\pi)^3}{(a_1, a_2, a_3)^3} (\vec{a}_1, \vec{a}_2, \vec{a}_3)^2 = \frac{(2\pi)^3}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)}$$

2.) $$\vec{b}_2 \times \vec{b}_3 = \frac{(2\pi)^2}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)^2} \underline{(\vec{a}_3 \times \vec{a}_1) \times (\vec{a}_1 \times \vec{a}_2)}$$

$$= \frac{(2\pi)^2}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)^{\cancel{2}}} \cancel{(\vec{a}_1, \vec{a}_2, \vec{a}_3)} \cdot \vec{a}_1 = \frac{(2\pi)^2}{(\vec{a}_1, \vec{a}_2, \vec{a}_3)} \vec{a}_1$$

$$\underline{\underline{\vec{C}_1}} = \frac{\cancel{(2\pi)^2}\,\vec{a}_1}{\cancel{(\vec{a}_1,\vec{a}_2,\vec{a}_3)}} \cdot \frac{\cancel{(\vec{a}_1,\vec{a}_2,\vec{a}_3)}}{\cancel{(2\pi)^3}}\,\cancel{2\pi} =$$

$$= \underline{\underline{\vec{a}_1}}$$



$\boxed{60 \to 8.10}$

$d = 0.405\ \text{nm}$

$\lambda = 0.3\ \text{nm}$

$$l = d\cos\varphi = d\cos\left(\frac{\pi}{2}-\Theta\right) = d\sin(\Theta)$$

$$2d\sin(\Theta) = n\lambda$$

$$\Theta \;\arcsin\left(\frac{n\lambda}{2d}\right)$$

$n=1$ 21.7°

$n=2$ 47.8°

$n=3$ X Ni rešitve

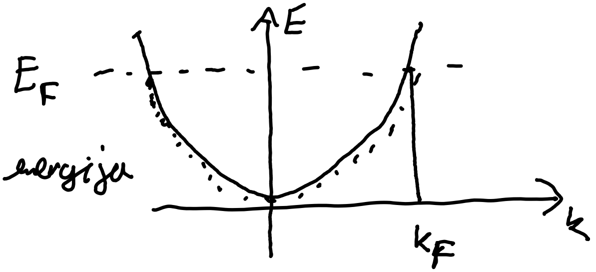
⋮

- - - - - - - - - - - - - - - - - - - -

Kovine in polprevodniki

Fermijeva energija $E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

$E = \frac{\hbar^2 k^2}{2m}$



E_F .. Fermijeva energija

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$

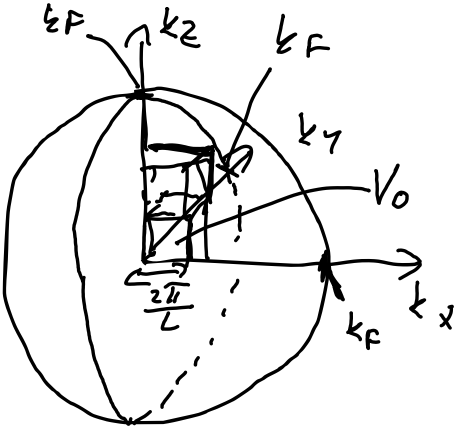
$\boxed{61} \rightarrow 9.1$ $\boxed{Cu}$ $E_F = 7eV$

m_e .. prosti e^-

$\rho = 8.96 \frac{kg}{dm^3}$

št. valenčnih e^- $Z = 1$

$A = 63,55$



$$E_F = \frac{\hbar^2 k_F^2}{2m_e}$$

$$k_F = \sqrt{\frac{2m_e E_F}{\hbar^2}}$$

$$V = \frac{4\pi k_F^3}{3}$$

$$k_{n_1 n_2 n_3} = \frac{2\pi}{L}(n_1, n_2, n_3)$$

$$V_0 = \left(\frac{2\pi}{L}\right)^3$$

N... # vseh e^- stanj

$$N = \underbrace{2}_{spin} \cdot \frac{V}{V_0} = 2 \cdot \frac{4\pi k_F^3}{3 \cdot \left(\frac{2\pi}{L}\right)^3} = \cancel{2}\frac{\cancel{4\pi} k_F^3 \cdot L^3}{3 \cdot \underset{2}{\cancel{8}}\, \underset{\pi^2}{\cancel{\pi^3}}}$$

$$N = \frac{k_F^3 L^3}{3\pi^2} \qquad L^3 = V$$

$$k_F = 1{,}36 \cdot 10^{10} \frac{1}{m}$$

$$n = \frac{N}{V} = \frac{k_F^3}{3\pi^2} = 8.5 \cdot 10^{28} \frac{1}{m^3}$$

<u>Alternativa</u>

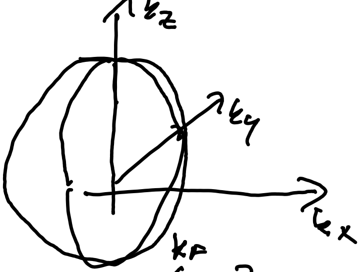
$$N = N_A \cdot n_{mol} \cdot Z = N_A \frac{m \cdot V}{M \cdot V} Z =$$

$$= N_A \frac{\rho \cdot V}{M} Z$$

$$n = \frac{N}{V} = \frac{N_A \cdot Z \cdot \rho}{M} = 8.5 \cdot 10^{28} \frac{1}{m^3}$$

$\boxed{\S 62 - 9.2}$

$$\frac{\overline{E}_{kin}}{E_F} = ?$$



$$\rightarrow N = \frac{2}{\left(\frac{2\pi}{L}\right)^3} \int dV = \frac{2 \cdot L^3}{(2\pi)^3} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_{-1}^{1} d(\cos\theta)}_{2} \cdot \int_0^{k_F} k^2 dk =$$

$$= \frac{2L^3}{(2\pi)^3} 4\pi \int_0^{k_F} k^2 dk = \frac{2L^3}{(2\pi)^3} 4\pi \frac{k_F^3}{3} = \frac{k_F^3 L^3}{3\pi^2}$$

$$\boxed{\overline{E}_{kin} = \frac{2}{(2\pi/L)^2} 4\pi \int_0^{k_F} dk\, k^2 \cdot \frac{\hbar^2 k^2}{2m}}$$

$$E_{kin} = \frac{L^3 \cancel{4\pi}}{\cancel{4}\pi^3} \int_0^{k_F} dk\, k^2 \frac{\hbar^2 k^2}{2m} =$$

$$= \frac{L^3}{\pi^2} \frac{\hbar^2}{2m} \int_0^{k_F} dk\, k^4 = \frac{\hbar^2 L^3}{\pi^2 2m} \frac{k_F^5}{5}$$

$$\bar{E}_{kin} = \frac{E_{kin}}{N} = \frac{\hbar^2 \cancel{L^3} \cdot \overset{k_F^2}{\cancel{k_F^5}} \; 3\cancel{\pi^2}}{\cancel{\pi^2} 2m \cdot 5 \; \cancel{L^3} \; \cancel{k_F^3}} = \frac{3 \hbar^2 k_F^2}{10\, m}$$

$$\frac{\bar{E}_{kin}}{E_F} = \frac{3 \cancel{\hbar^2} \cancel{k_F^2} \cdot 2 \cancel{m}}{10 \cancel{m} \; \cancel{\hbar^2} \cancel{k_F^2}} = \frac{3}{5}$$