Rešitve izpita 3.2.2023

1. naloga (25 točk)

Dani sta zaporedji s splošnima členoma

$$a_n = \frac{1+n}{n+n^2} \quad \text{in} \quad b_n = 2 \cdot \left(\frac{2}{3}\right)^{n-1}$$

a) (10 točk) Pokaži, da je zaporedje $(a_n)_{n\in\mathbb{N}}$ padajoče.

Pokazati moramo, da za vsak $m \in \mathbb{N}$ velja:

$$a_{m+1} \leq a_m .$$

$$\frac{1+m+1}{(m+1)+(m+1)^2} \leq \frac{1+m}{m+m^2}$$

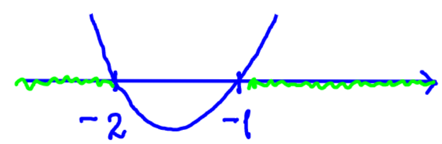
$$\frac{m+2}{m^2+3m+2} \leq \frac{1+m}{m+m^2}$$

$$(m+m^2)(m+2) \leq (1+m)(m^2+3m+2)$$

$$\cancel{m^3}+3m^2+2m \leq \cancel{m^3}+4m^2+5m+2$$

$$m^2+3m+2 \geq 0$$

$$(m+2)(m+1) \geq 0 \quad ✓$$



za vsak $m \in \mathbb{N}$

b) (5 točk) Izračunaj $\lim\limits_{n\to\infty} a_n$.

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{1+m \quad /:m^2}{m+m^2 \quad /:m^2} =$$

$$= \lim_{m\to\infty} \frac{\frac{1}{m^2} + \frac{1}{m}}{\frac{1}{m}+1} = \frac{0}{1} = \underline{\underline{0}}$$

$$\frac{1}{m^2} \to 0, \quad \frac{1}{m} \to 0, \quad \frac{1}{m} \to 0$$

c) (10 točk) Utemelji, da vrsta $\sum_{n=1}^{\infty} b_n$ konvergira in izračunaj njeno vsoto.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 2\cdot\left(\frac{2}{3}\right)^{n-1} = 2\cdot\left(\frac{2}{3}\right)^{0} + 2\cdot\left(\frac{2}{3}\right)^{1} + 2\cdot\left(\frac{2}{3}\right)^{2} + \ldots$$

$\cdot\frac{2}{3}$ $\quad\cdot\frac{2}{3}$ $\quad\ldots$

Dana vrsta konvergira, saj gre za geometrijsko vrsto s $q = \frac{2}{3}$.

Ker je $|q| < 1$, dana geometrijska vrsta konvergira.

$$\sum_{n=1}^{\infty} b_n = \frac{a_1}{1-q} = \frac{2}{1-\frac{2}{3}} = \frac{2}{\frac{1}{3}} = \underline{\underline{6}}$$

2. naloga (25 točk)

Dana je funkcija $f(x) = x\sqrt{x+6}$.

a) (10 točk) Določi definicijsko območje ter ničle funkcije f.

- D_f:

$$x + 6 \geq 0$$
$$x \geq -6$$
$$x \in [-6, \infty)$$

- ničle:

$$x\sqrt{x+6} = 0$$
$$\boxed{x_1 = 0} \quad \boxed{x_2 = -6}$$

b) (10 točk) Določi lokalne ekstreme ter zapiši intervale naraščanja in padanja funkcije f.

Stacionarne točke: $f'(x) = 0$.

$$f'(x) = (x)' \cdot \sqrt{x+6} + x \cdot \left(\sqrt{x+6}\right)' =$$

$$= 1 \cdot \sqrt{x+6} + \frac{x}{2\sqrt{x+6}} \cdot (x+6)' =$$

$$= \sqrt{x+6} + \frac{x}{2\sqrt{x+6}} \cdot 1 =$$

$$= \frac{2\left(\sqrt{x+6}\right)^2 + x}{2\sqrt{x+6}} =$$

$$= \frac{2(x+6)+x}{2\sqrt{x+6}} =$$

$$= \frac{3x+12}{2\sqrt{x+6}} = 0$$

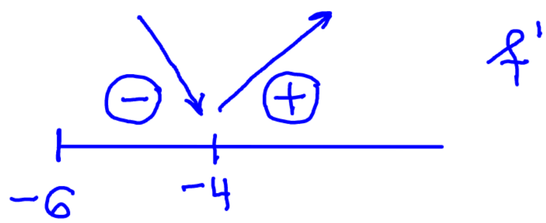
$$3x+12=0$$

$$\boxed{x=-4}$$

$T(-4, -4\sqrt{2})$

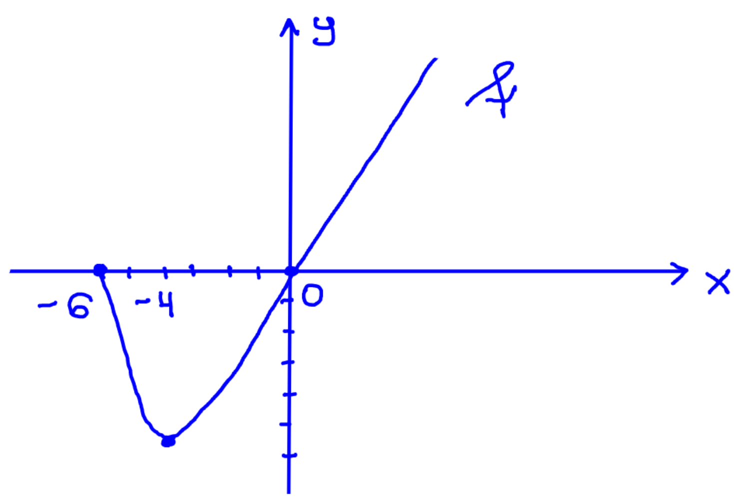
lokalni minimum

Naraščanje, padanje:



f narašča na $(-4, \infty)$ in pada na $[-6, -4)$

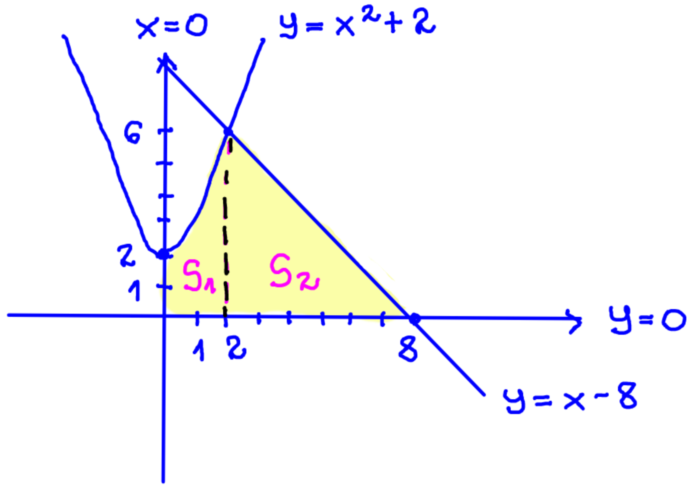
c) (5 točk) Skiciraj graf funkcije f.



3. naloga (25 točk)

Izračunaj ploščino lika, ki ga omejujejo krivulje $y = x^2 + 2$, $y = -x + 8$, $x = 0$ in $y = 0$. Nariši skico območja.

Skica:



Presečišča med krivuljama $y = x^2+2$ in $y=-x+8$:

$$x^2+2=-x+8$$
$$x^2+x-6=0$$
$$(x+3)(x-2)=0$$
$$x_1=-3 \quad x_2=2$$

Iščemo ploščino rumeno obarvanega območja:

$$S = S_1 + S_2 = \int_0^2 (x^2+2)\,dx + \int_2^8 (-x+8)\,dx =$$

$$= \left(\frac{x^3}{3}+2x\right)\Big|_0^2 + \left(-\frac{x^2}{2}+8x\right)\Big|_2^8 =$$

$$= \frac{2^3}{3} + 2\cdot 2 + \left(-\frac{8^2}{2}+8\cdot 8\right) - \left(-\frac{2^2}{2}+8\cdot 2\right) =$$

$$= \frac{8}{3} + 4 - 32 + 64 + 2 - 16 =$$

$$= \frac{8}{3} + 22 = \frac{8+66}{3} = \underline{\underline{\frac{74}{3}}}$$

4. naloga (25 točk)

a) (10 točk) Dani sta matriki

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -6 \\ 2 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 3 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

Izračunaj matriko $A \cdot B^T - I_3$.

$$A \cdot B^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -6 \\ 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 2 \\ -2 & 3 & 0 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 4 \\ -22 & 22 & -4 \\ 12 & -5 & 6 \end{bmatrix}$$

$$A \cdot B^T - I_3 = \begin{bmatrix} 5 & 0 & 4 \\ -22 & 21 & -4 \\ 12 & -5 & 5 \end{bmatrix}$$

b) (15 točk) Reši sistem linearnih enačb:

$$\begin{aligned} x &+ y &+ 2z &= -1 \\ x &+ 3y &- 6z &= 7 \\ 2x &- y &+ 2z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 3 & -6 & 7 \\ 2 & -1 & 2 & 0 \end{array}\right] /\cdot(-2) \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -2 & 8 & -8 \\ 0 & -3 & -2 & 2 \end{array}\right] \begin{array}{l} \\ /\cdot(-3) \\ /\cdot 2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -2 & 8 & -8 \\ 0 & 0 & -28 & 28 \end{array}\right] \begin{array}{l} \\ /:2 \\ /:28 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & -4 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$-z = 1$

$\boxed{z = -1}$

$-y + 4z = -4$

$-y = -4 - 4(-1)$

$\boxed{y = 0}$

$x + y + 2z = -1$

$x = -1 - 0 - 2(-1)$

$\boxed{x = 1}$

$\boxed{T(1, 0, -1)}$