Ime in priimek

Vpisna številka

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Osnove matematične analize: drugi kolokvij

11. januar 2023

Čas pisanja je 90 minut. Dovoljena je uporaba 1 lista A4 formata s formulami. Uporaba kalkulatorja ali drugih pripomočkov ni dovoljena. Vse odgovore dobro utemelji!

1. naloga (25 točk)

Naj bo

$$f(x) = \sqrt{x^2 - x + 1}\ .$$

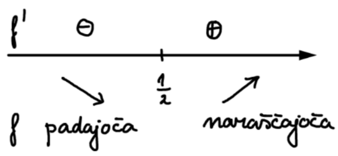
a) (8 točk) Poišči stacionarne točke funkcije f.

$$f'(x) = \frac{1}{\underbrace{2\sqrt{x^2-x+1}}} \cdot (2x-1) = 0 \longrightarrow 2x-1=0 \longrightarrow \underline{\underline{x=\tfrac{1}{2}}} \qquad f(\tfrac12) = \sqrt{\tfrac14 - \tfrac12 + 1} = \sqrt{\tfrac34}$$

$D = 1-4\cdot1\cdot1 < 0$

$\Rightarrow x^2-x+1>0$

$\Rightarrow 2\sqrt{x^2-x+1} > 0$



$\Rightarrow x = \frac12$ je minimum

b) (8 točk) Kje je funkcija f konveksna in kje konkavna?

$$f''(x) = \frac{2\cdot2\sqrt{x^2-x+1} - (2x-1)\cdot 2\,\frac{1}{2\sqrt{x^2-x+1}}(2x-1)}{\underbrace{\left(2\sqrt{x^2-x+1}\right)^2}_{>0}} = \frac{4\sqrt{x^2-x+1} - \frac{(2x-1)^2}{\sqrt{x^2-x+1}}}{\oplus} = \frac{\frac{4(x^2-x+1)-(2x-1)^2}{\sqrt{x^2-x+1}\ \oplus}}{\oplus} =$$

$$= \frac{4x^2-4x+4-4x^2+4x-1}{\oplus} = \frac{3}{\oplus} > 0 \Rightarrow \underline{\underline{f \text{ je konveksna na } \mathbb{R}}}$$

c) (9 točk) Poišči točko na grafu funkcije f, ki je najmanj oddaljena od točke $T(1,0)$.

d bo minimalna $\Leftrightarrow$ d^2 bo minimalna $\qquad (\sqrt{x})^2 = x$

$$d^2\big((x, f(x)), (1,0)\big) = (x-1)^2 + (f(x)-0)^2 = (x-1)^2 + \left(\sqrt{x^2-x+1}\right)^2 = x^2-2x+1+x^2-x+1 =$$

$$= 2x^2-3x+2 = D(x)$$

$$D'(x) = 4x-3 = 0$$

$$\underline{\underline{x = \tfrac34}}$$

$$y = f(\tfrac34) = \sqrt{(\tfrac34)^2 - \tfrac34 + 1} = \sqrt{\tfrac{9}{16} + \tfrac14} = \sqrt{\tfrac{9+4}{16}} = \tfrac{\sqrt{13}}{4}$$

$$\underline{\underline{T\left(\tfrac34, \tfrac{\sqrt{13}}{4}\right)}}$$

2. naloga (25 točk)

Naj bo
$$f(x,y) = \log y - \frac{9}{2}y^2 - 6x^2y.$$

a) (7 točk) Izračunaj smerni odvod funkcije f v točki $(1,1)$ v smeri najhitrejšega naraščanja.

$$\left.\begin{aligned} f_x(x,y) &= -12xy \\ f_y(x,y) &= \frac{1}{y} - 9y - 6x^2 \end{aligned}\right\} \quad \begin{aligned} (\operatorname{grad} f)(x,y) &= \left(-12xy,\ \frac{1}{y} - 9y - 6x^2\right) \\ (\operatorname{grad} f)(1,1) &= (-12,-14) \ = \text{smer najhit. nar.} \end{aligned}$$

$$f_{(-12,-14)}(1,1) = \frac{(-12,-14)\cdot(-12,-14)}{\|(-12,-14)\|} = \frac{(-12)^2 + (-14)^2}{\sqrt{(-12)^2 + (-14)^2}} = \frac{144+196}{\sqrt{144+196}} = \frac{340}{\sqrt{340}} = \underline{\underline{\sqrt{340}}}$$

b) (9 točk) Določi vse stacionarne točke funkcije f.

$$\begin{aligned} f_x = 0 &\rightsquigarrow -12xy = 0 \longrightarrow xy = 0 \longrightarrow y = 0 \\ f_y = 0 &\rightsquigarrow \underline{\frac{1}{y} - 9y - 6x^2 = 0} \end{aligned}$$

$y = 0$: $\mathrm{II}: \times$

$x = 0$

$\mathrm{II}: \frac{1}{y} - 9y = 0$

$\frac{1}{y} = 9y$

$y^2 = \frac{1}{9}$

$y = \pm\frac{1}{3}$ $\quad \underline{\underline{T_1\left(0,\frac{1}{3}\right)}}$ ✓ $\quad \underline{\underline{T_2\left(0,-\frac{1}{3}\right)}} \notin D_f = \mathbb{R}\times\mathbb{R}^+$

c) (9 točk) Klasificiraj vse stacionarne točke funkcije f.

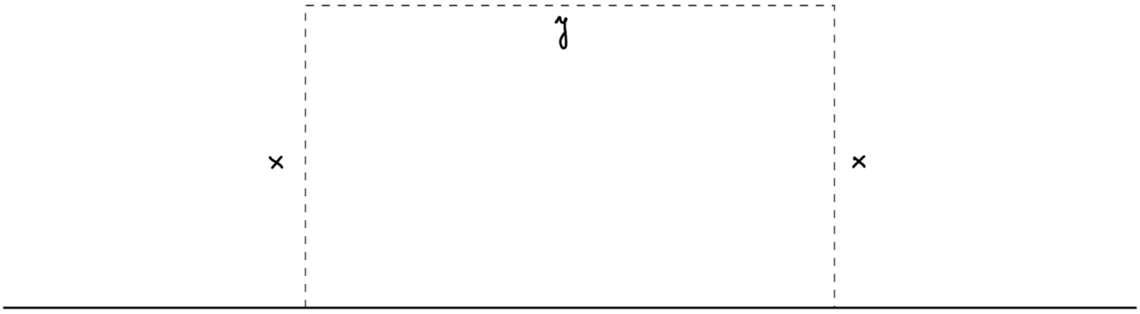
$$\left.\begin{aligned} f_{xx} &= -12y \\ f_{xy} &= -12x \\ f_{yy} &= -\frac{1}{y^2} - 9 \end{aligned}\right\} \quad H_f = \begin{bmatrix} -12y & -12x \\ -12x & -\frac{1}{y^2} - 9 \end{bmatrix}$$

- $T_1\left(0,\frac{1}{3}\right)$ $\quad H_f\left(0,\frac{1}{3}\right) = \begin{bmatrix} -4 & 0 \\ 0 & -18 \end{bmatrix}$ $\quad \det H_f\left(0,\frac{1}{3}\right) = -4\cdot(-18) > 0$ $\quad \underline{\underline{T_1 \text{ je maksimum}}}$

(• $T_2\left(0,-\frac{1}{3}\right)$ $\quad H_f\left(0,-\frac{1}{3}\right) = \begin{bmatrix} 4 & 0 \\ 0 & -18 \end{bmatrix}$ $\quad \det H_f\left(0,-\frac{1}{3}\right) = 4\cdot(-18) < 0$ $\quad \underline{\underline{T_2 \text{ je sedlo}}}$)

3. naloga (25 točk)

Ob dolgem ravnem zidu hočemo postaviti ograjo pravokotne oblike dolžine ℓ kot je narisano na spodnji sliki. Zid je označen s polno črto, ograja, ki jo hočemo postaviti, pa s črtkano črto (postaviti moramo samo tri stranice pravokotne ograje).



Kakšne morajo biti dimenzije pravokotne ograje, da bo površina, ki jo bomo zagradili, največja možna (glede na dano dolžino ograje ℓ)?

$S(x,y) = xy$ vez: $\ell = 2x + y \rightsquigarrow y = \ell - 2x$

$S(x) = x(\ell - 2x) = x\ell - 2x^2$

$S'(x) = \ell - 4x = 0$

$4x = \ell$

$\underline{\underline{x = \frac{\ell}{4}}}$ $y = \ell - 2\cdot\frac{\ell}{4} = \ell - \frac{\ell}{2}$

$\underline{\underline{y = \frac{\ell}{2}}}$

$\Rightarrow$ Del ograje, ki je vzporeden zidu, mora meriti $\frac{\ell}{2}$, dela ograje, ki sta pravokotna na zid, pa vsak po $\frac{\ell}{4}$.

2. način (vezani ekstrem) $F(x,y,\lambda) = S(x,y) - \lambda(2x + y - \ell) = xy - 2\lambda x - \lambda y + \lambda\ell$

$F_x = y - 2\lambda = 0 \longrightarrow y = 2\lambda$

$F_y = x - \lambda = 0 \longrightarrow x = \lambda$

$F_\lambda = 2x + y - \ell = 0$

$2\lambda + 2\lambda = \ell$

$4\lambda = \ell$

$\lambda = \frac{\ell}{4} \longrightarrow \underline{\underline{x = \frac{\ell}{4}, \ y = \frac{\ell}{2}}}$

4. naloga (25 točk)

Izračunaj naslednja nedoločena integrala.

a) (10 točk)

$$I_1 = \int \frac{\sqrt{\arctan(x)}}{1+x^2}\,dx.$$

Spomnimo se: $(\mathrm{arctg}\,x)' = \frac{1}{1+x^2}$

$$I_1 = \int \sqrt{t}\,dt = \int t^{\frac{1}{2}}\,dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \underline{\underline{\frac{2}{3}(\mathrm{arctg}\,x)^{\frac{3}{2}} + c}}$$

$t = \mathrm{arctg}\,x$

$dt = \frac{1}{1+x^2}\,dx$

b) (15 točk)

$$\int \frac{x}{\cos^2 x}\,dx.$$

Spomnimo se: $(\mathrm{tg}\,x)' = \frac{1}{\cos^2 x}$. Torej je $\int \frac{1}{\cos^2 x}\,dx = \mathrm{tg}\,x + c$.

$u = x \rightsquigarrow du = dx$

$dv = \frac{1}{\cos^2 x}\,dx \rightsquigarrow v = \mathrm{tg}\,x$

$$I_1 = \int u\,dv = uv - \int v\,du = x\,\mathrm{tg}\,x - \int \mathrm{tg}\,x\,dx = \underline{\underline{x\,\mathrm{tg}\,x + \log|\cos x| + c}}$$

$$\int \mathrm{tg}\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{dt}{t} = -\log|t| + c = -\log|\cos x| + c$$

$t = \cos x$

$dt = -\sin x\,dx$