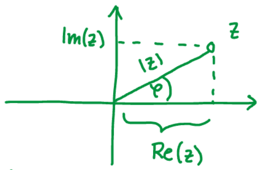
Osnove matematične analize
Vaje, 3. teden

1. Število $z = \frac{1+3i}{1-i}$ zapiši v obliki $x + iy$ in izračunaj $|z|$ ter $\arg(z)$.

$$z = \underbrace{\mathrm{Re}(z)}_{\in \mathbb{R}} + i\,\underbrace{\mathrm{Im}(z)}_{\in \mathbb{R}} \in \mathbb{C}$$

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$

$$\arg(z) = \varphi = \mathrm{arctg}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$$



$$z = \frac{1+3i}{1-i} = \frac{(1+3i)(1+i)}{(1-i)(1+i)} = \frac{1+i+3i-3}{1+1} =$$

$$(a+b)(a-b) = a^2 - b^2$$

$$(a+ib)(a-ib) = a^2 + b^2$$

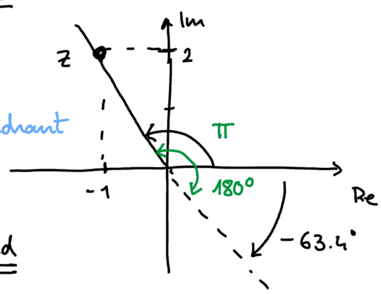
$$= \frac{-2+4i}{2} = \underline{\underline{-1+2i}}$$

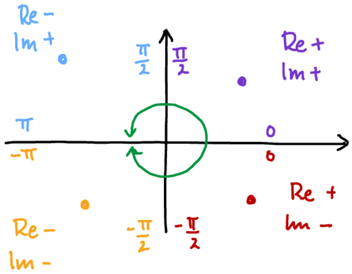
$$|z| = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \underline{\underline{\sqrt{5}}}$$

popravek za kvadrant

$$\arg(z) = \mathrm{arctg}\left(\frac{2}{-1}\right) = \mathrm{arctg}(-2) = -63.4^\circ + 180^\circ = \underline{\underline{116.6^\circ}}$$

$$= -1.1 \text{ rad} + \pi = \underline{\underline{2.04 \text{ rad}}}$$





$$\mathrm{atan2}(y, x) = \mathrm{atan2}(\mathrm{Im}(z), \mathrm{Re}(z)) \in [-\pi, \pi]$$

2. * Nariši množico točk:

(a) $|\overline{z}+2-i|\le 2$,

(b) $\mathrm{Re}(\overline{z}+2-i)\le 2$,

(c) $\mathrm{Im}(\overline{z}+2-i)\le 2$.

$|z| = \sqrt{\mathrm{Re}(z)^2+\mathrm{Im}(z)^2}$

$$\boxed{(x-p)^2+(y-q)^2=r^2}$$

množica z radijem r in središčem $S(p,q)$

$$\boxed{|z-(p+iq)|=r}$$

$|\overline{z}+2-i|\le 2$

$z = x+iy$

a) $|x-iy+2-i|\le 2$

$|x+2-i(y+1)|\le 2$

$\sqrt{(x+2)^2+(-(y+1))^2}\le 2$

$(x+2)^2+(y+1)^2\le 4$

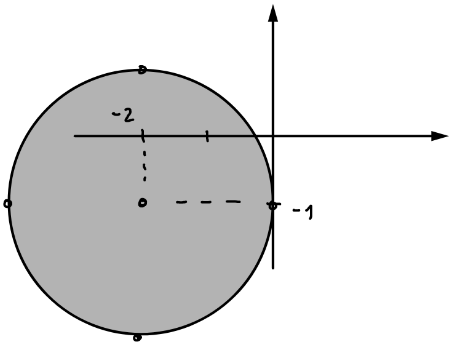
$\mathrm{Re}(\overline{z}+2-i) = x+2$

$\mathrm{Im}(\overline{z}+2-i) = -(y+1)$

krog s središčem $S(-2,-1)$ in radijem 2

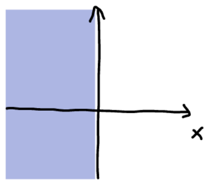
$\downarrow$

$-2-i$



b) $x+2\le 2$

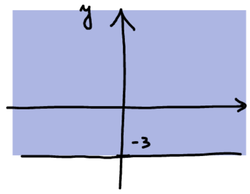
$x\le 0$



c) $-(y+1)\le 2 \quad |\cdot(-1)$

$y+1\ge -2$

$y\ge -3$



3. * Reši naslednje enačbe:

(a) $z^2 + z = 1$,

$z = x + iy$, $\boxed{\begin{matrix} x \in \mathbb{R} \\ y \in \mathbb{R} \end{matrix}}$

$$(x+iy)^2 + (x+iy) = 1$$

$$x^2 + 2xyi - y^2 + x + iy = 1 + 0 \cdot i$$

Re : $x^2 - y^2 + x = 1$ *

Im : $2xy + y = 0 \longrightarrow y(2x+1) = 0$

$y = 0$:

* $x^2 + x = 1$

$x^2 + x - 1 = 0$

$D = 1 - 4 \cdot 1 \cdot (-1) = 5$

$x_{1,2} = \dfrac{-1 \pm \sqrt{5}}{2}$

$z_1 = \dfrac{-1+\sqrt{5}}{2} + 0 \cdot i$, $z_2 = \dfrac{-1-\sqrt{5}}{2} + 0 \cdot i$

edini rešitvi

$2x + 1 = 0$:

$x = -\frac{1}{2}$

* $\left(-\frac{1}{2}\right)^2 - y^2 + \left(-\frac{1}{2}\right) = 1$

$\frac{1}{4} - y^2 - \frac{1}{2} = 1$

$-y^2 = \frac{5}{4}$

$y^2 = -\frac{5}{4}$

$y = \pm i \cdot \frac{\sqrt{5}}{2}$ //

ni IR rešitev

(b) $(2+i)z + 2z - 3 = 4 + 6i$.

$z = x + iy$

$$(2+i)(x+iy) + 2(x+iy) - 3 = 4 + 6i$$

$$2x + ix + 2yi - y + 2x + 2yi - 3 = 4 + 6i$$

Re : $4x - y = 7$ $\quad / \cdot 4$

Im : $4y + x = 6$ $\quad)+$

$16x - 4y = 28$

$17x = 34$

$x = 2$

$4 \cdot 2 - y = 7$

$y = 1$

$z = 2 + i$

2. način

$(4+i)z - 3 = 4 + 6i$

$(4+i)z = 7 + 6i \quad / : (4+i)$

$z = \dfrac{7+6i}{4+i}$

$$z = \frac{7+6i}{4+i} = \frac{(7+6i)(4-i)}{(4+i)(4-i)} = \frac{28 - 7i + 24i + 6}{16 + 1} = \frac{34 + 17i}{17} = 2 + i$$

4. * Reši naslednje enačbe:

(a) $2z^2 - 3\overline{z}^2 = 10i$,

$z = x + iy$ $\longrightarrow$ $2(x+iy)^2 - 3(x-iy)^2 = 10i$

$x \in \mathbb{R}$
$y \in \mathbb{R}$

$2(x^2 + 2xyi - y^2) - 3(x^2 - 2xyi - y^2) = 10i$

$2x^2 + 4xyi - 2y^2 - 3x^2 + 6xyi + 3y^2 = 10i + 0$

Re: $-x^2 + y^2 = 0$

Im: $10xy = 10 \longrightarrow xy = 1 \longrightarrow \boxed{y = \frac{1}{x}} \longrightarrow -x^2 + \left(\frac{1}{x}\right)^2 = 0$

$\frac{1}{x^2} = x^2 \quad / \cdot x^2$

$x^4 = 1$

$1 \quad -1 \quad i \quad -i$

$x_1 = 1$, $y_1 = 1$, $z_1 = 1 + i$

$x_2 = -1$, $y_2 = -1$, $z_2 = -1 - i$

$z = \pm 1 \pm i$

(b) $\overline{z} - iz^2 = 0$.

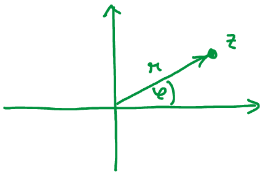
5. Z uporabo polarne oblike in de Moivrove formule izračunaj

(a) * $\left(-\frac{1}{2}+\frac{i}{2}\right)^8$,

(b) * $\left(-1-i\sqrt{3}\right)^{20}$,

(c) $(1-i)^{5000}$,

(d) $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$.



$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$

$|z| = r$ ... radij

$\arg(z) = \varphi$ ... polarni kot

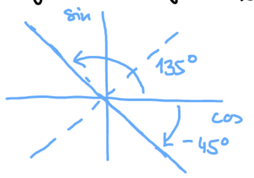
$$\boxed{\begin{array}{l} z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi} \\ z^n = r^n\left(\cos(n\varphi)+i\sin(n\varphi)\right) = r^n\cdot e^{in\varphi}\end{array}}$$

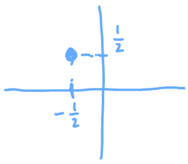
$|z| = \sqrt{\mathrm{Re}(z)^2+\mathrm{Im}(z)^2}$ $\quad \arg(z) = \mathrm{arctg}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$

(a) * $\underbrace{\left(-\frac{1}{2}+\frac{i}{2}\right)}_{z}{}^8$

$$r = |z| = \sqrt{\left(-\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}} = \sqrt{\tfrac{1}{2}} = \underline{\underline{\tfrac{1}{\sqrt{2}}}}$$

$$\arg(z) = \mathrm{arctg}\left(\frac{\frac{1}{2}}{-\frac{1}{2}}\right) = \mathrm{arctg}(-1) = -45^\circ + 180^\circ = 135^\circ = \underline{\underline{\tfrac{3\pi}{4}}}$$

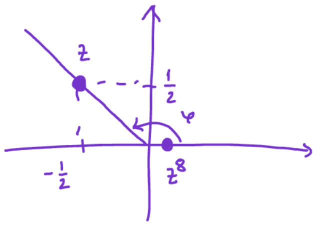


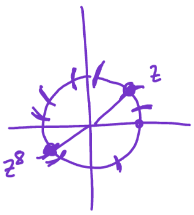


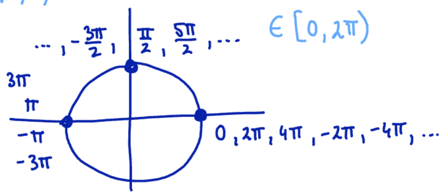
$$z = \frac{1}{\sqrt{2}}\,e^{i\cdot\frac{3\pi}{4}}$$

$$z^8 = \underbrace{\left(\tfrac{1}{\sqrt{2}}\right)^8}_{\left(\left(\frac{1}{\sqrt{2}}\right)^2\right)^4}\cdot e^{i\cdot 8\cdot\frac{3\pi}{4}} = \frac{1}{16}e^{i\,6\pi} = \frac{1}{16}\underbrace{e^{i\cdot 0}}_{1} = \underline{\underline{\frac{1}{16}}}$$

$\in [0, 2\pi)$







$$6\pi \xrightarrow{-2\pi} 4\pi \xrightarrow{-2\pi} 2\pi \xrightarrow{-2\pi} 0$$

$-3\cdot 2\pi$

(b) * $\underbrace{\left(-1-i\sqrt{3}\right)}_{z}{}^{20}$

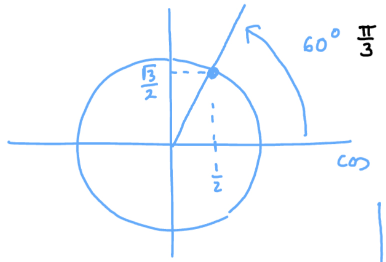
$$r = |z| = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

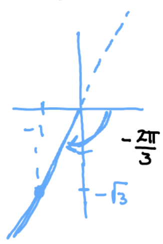
$$\varphi = \arg(z) = \mathrm{arctg}\left(\frac{-\sqrt{3}}{-1}\right) = \mathrm{arctg}(\sqrt{3}) = \mathrm{arctg}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$$

sin, cos

$= 60^\circ \pm 180^\circ \to 240^\circ,\ -120^\circ$

$= \frac{\pi}{3} \pm \pi \to \frac{4\pi}{3},\ -\frac{2\pi}{3}$





$6\cdot 13$

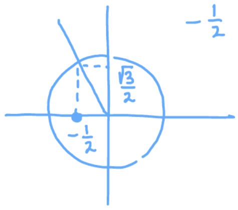
$80 = 78+2$

$$z = 2e^{i\cdot\frac{4\pi}{3}}$$

$$z^{20} = 2^{20}e^{i\cdot 20\cdot\frac{4\pi}{3}} = 2^{20}\cdot e^{i\cdot\frac{80\pi}{3}} = \underline{\underline{2^{20}\cdot e^{i\cdot\frac{2\pi}{3}}}}$$

$$\frac{80\pi}{3} = \frac{(6\cdot 13+2)\pi}{3} = \underline{\underline{13\cdot 2\pi}} + \frac{2\pi}{3}$$

$$z^{20} = 2^{20}\cdot\Big(\underbrace{\cos\left(\tfrac{2\pi}{3}\right)}_{-\frac{1}{2}} + i\underbrace{\sin\left(\tfrac{2\pi}{3}\right)}_{\frac{\sqrt{3}}{2}}\Big) = 2^{20}\left(-\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right) = \underline{\underline{2^{19}\left(-1+i\sqrt{3}\right)}}$$



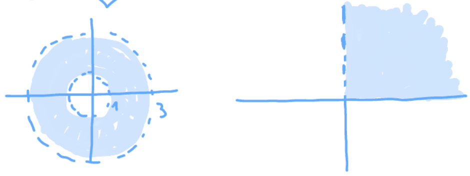
8. Nariši naslednje podmnožice v $\mathbb{C}$:

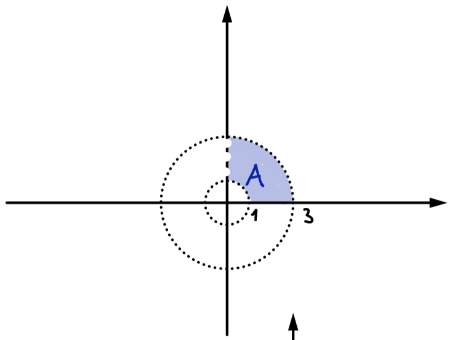
$$A = \{z \in \mathbb{C}\,;\ 1 < |z| < 3, 0 \leq Arg(z) < \pi/2\},$$

$$B = \{z \in \mathbb{C}\,;\ 1/3 < |z-1| < 3, 0 \leq Arg(z-1) < \pi\},$$
$$C = \{z \in \mathbb{C}\,;\ 1 < |z| < 3, \pi < Arg(z) \leq 3\pi/2\}.$$

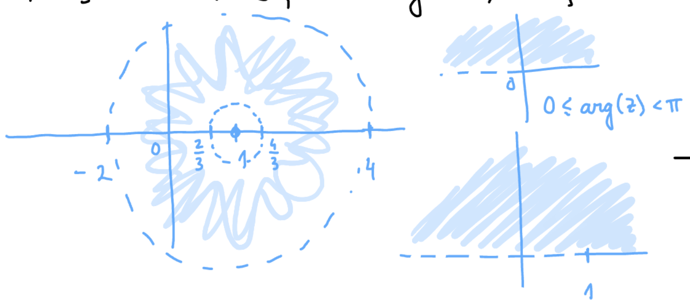
Nato poišči kompleksne transformacije, ki transformirajo A v B, A v C in C v A.

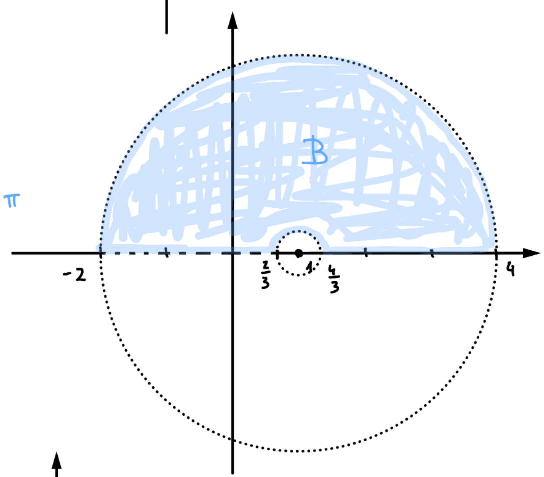
$A = \{z \in \mathbb{C}\,;\ \underbrace{1 < |z| < 3}\,,\ 0 \leq \arg(z) < \frac{\pi}{2}\}$



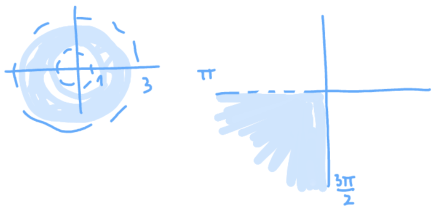


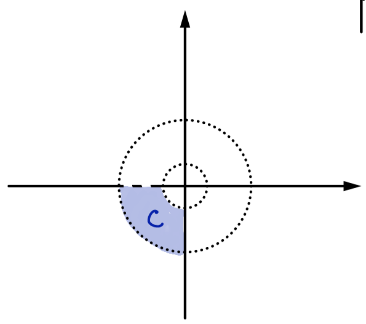
$B = \{z \in \mathbb{C}\,;\ \frac{1}{3} < |z-1| < 3,\ 0 \leq \arg(z-1) < \pi\}$





$C = \{z \in \mathbb{C}\,;\ 1 < |z| < 3,\ \pi < \arg(z) \leq \frac{3\pi}{2}\}$





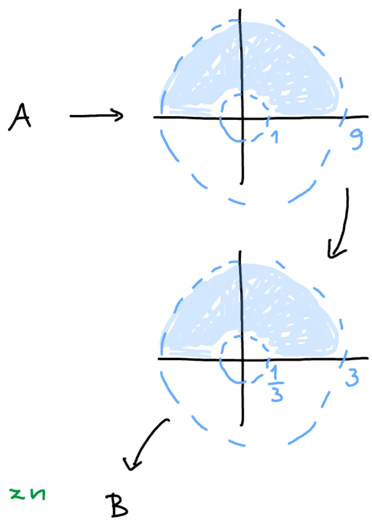
Preslikava iz A v B?

$z \longmapsto z^2$ koti med 0 in $\frac{\pi}{2}$ $\longrightarrow$ med 0 in π

radij med 1 in 3 $\longrightarrow$ med 1 in 9

$z^2 \longmapsto \frac{z^2}{3}$ radij med 1 in 9 $\longrightarrow$ radij med $\frac{1}{3}$ in 3

$\frac{z^2}{3} \longmapsto \frac{z^2}{3}+1$ središče kolobarja v 0 $\longrightarrow$ središče v 1



$z \longmapsto z^n$ ... radij se potencira z n
polarni kot se pomnoži z n

$f_{AB}: A \longrightarrow B$

$z \longmapsto \frac{z^2}{3}+1$

$z \longmapsto \frac{z}{a}$ $(a \in \mathbb{R})$ ... razteg s središčem v 0 s faktorjem $\frac{1}{a}$

$z \longmapsto az$ a

$z \longmapsto z + a + bi \quad (a, b \in \mathbb{R}^{+})$ ... premik (translacija) za a desno in b gor