Osnove matematične analize
Vaje 4

1. * Dokaži, da je zaporedje, podano kot

$$a_0 = 0,\ a_{n+1} = \frac{-2}{a_n + 3}$$

konvergentno in izračunaj njegovo limito.

$a_0 = 0 \qquad a_1 = \frac{-2}{0+3} = -\frac{2}{3} \qquad a_2 = \frac{-2}{-\frac{2}{3}+3} = \frac{-6}{-2+9} = -\frac{6}{7}$

- a_n je padajoče z indukcijo: $a_{n+1} < a_n \quad \forall n \in \mathbb{N}$

Baza: $n=0$ $\quad a_1 < a_0$

$-\frac{2}{3} < 0$ ✓

Indukcijski korak: $n \to n+1$

vemo: $a_n < a_{n-1}$ $\longrightarrow$

radi bi videli: $a_{n+1} < a_n$

$a_n < a_{n-1} \quad | +3$

$a_n + 3 < a_{n-1} + 3 \quad | \,\hat{}\,(-1)$

$\frac{1}{a_n+3} > \frac{1}{a_{n-1}+3} \quad | \cdot(-2)$

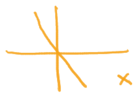
$\frac{-2}{a_n+3} < \frac{-2}{a_{n-1}+3}$

$\underline{\underline{a_{n+1} < a_n}}$

$a_{n+1} = \frac{-2}{a_n+3}$

$x \mapsto \frac{1}{x}$

padajoča



$x \mapsto -2x$

padajoča

- a_n je navzdol omejeno z indukcijo: $a_n > m \quad \forall n \in \mathbb{N}$

$m = -1$ $\qquad$ Baza: $n=0$ $\quad a_0 > -1$

$0 > -1$ ✓

Indukcijski korak: $n \to n+1$

vemo: $a_n > -1$ $\longrightarrow$

radi bi videli: $a_{n+1} > -1$

$a_n > -1 \quad | +3$

$a_n + 3 > 2 \quad | \,\hat{}\,(-1)$

$\frac{1}{a_n+3} < \frac{1}{2} \quad | \cdot(-2)$

$\frac{-2}{a_n+3} > -1$

$\underline{\underline{a_{n+1} > -1}}$

Ker je $\{a_n\}$ padajoče in navzdol omejeno, je konvergentno. Naj bo $a = \lim\limits_{n\to\infty} a_n$.

$$a_{n+1} = \frac{-2}{a_n+3}$$

$\downarrow_{n\to\infty} \qquad \downarrow_{n\to\infty}$

$$a = \frac{-2}{a+3}$$

$$a(a+3) = -2$$

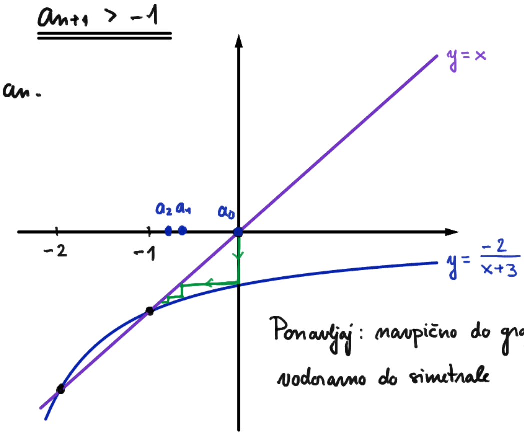
$a^2 + 3a + 2 = 0$

$(a+1)(a+2) = 0$

$a_1 = -1 \quad a_2 = -2$

kandidata za lim

Ker je $a_n > -1$, je $a = -1$.



Ponavljaj: navpično do grafa, vodoravno do simetrale

2. Dokaži, da je zaporedje, podano kot

$$a_0 = -3, \ a_{n+1} = e^{a_n} - 1$$

konvergentno in izračunaj njegovo limito. Kaj pa če vzamemo za prvi člen zaporedja $a_0 = 3$?

$a_0 = -3 \qquad a_1 = e^{-3} - 1 = -0.95 \qquad a_2 = \cdots$

- a_n je naraščajoče: $a_{n+1} > a_n$ za $\forall n \in \mathbb{N}$

Baza: $n = 0$

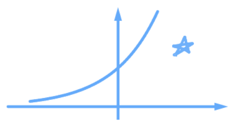
$$a_1 > a_0$$
$$e^{-3} - 1 > -3$$
$$e^{-3} > -2 \ \checkmark \quad (\text{ker } e^{-3} > 0)$$

Ind. korak: vemo: $a_n > a_{n-1}$

radi bi videli: $a_{n+1} > a_n$

$$a_n > a_{n-1} \quad / \exp$$
$$e^{a_n} > e^{a_{n-1}} \quad / -1$$
$$e^{a_n} - 1 > e^{a_{n-1}} - 1$$
$$\underline{\underline{a_{n+1} > a_n}}$$

exp naraščajoča



- a_n je navzgor omejeno z $M = 0$: $a_n < 0 \ \forall n \in \mathbb{N}$

Baza: $n = 0$

$$a_0 < 0$$
$$-3 < 0 \ \checkmark$$

Ind. korak: radi bi videli: $a_{n+1} < 0$

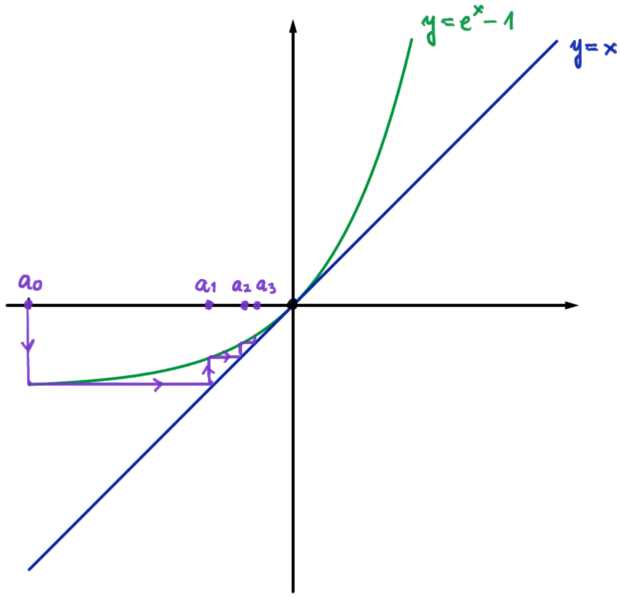
vemo: $a_n < 0$

$$a_n < 0 \quad / \exp$$
$$e^{a_n} < e^0 \quad / -1$$
$$e^{a_n} - 1 < 1 - 1$$
$$\underline{\underline{a_{n+1} < 0}}$$

Ker je a_n naraščajoče in navzgor omejeno, je konvergentno. $a = \lim_{n \to \infty} a_n$.

$$a_{n+1} = e^{a_n} - 1$$
$$\downarrow n \to \infty \qquad \downarrow n \to \infty$$
$$a = e^a - 1$$

Uganemo rešitev: $a = 0$.



3. * Izračunaj limite naslednjih zaporedij:

$$a)\ \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} \qquad b)\ \lim_{n\to\infty}\left(2-\frac{n}{n+1}\right)^{2n-1}$$

$$c)\ \lim_{n\to\infty} n\left(\log(n+1)-\log n\right) \qquad d)\ \lim_{n\to\infty}\frac{n^2\sin(n^2)+n-6}{n^3-n+1}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \qquad \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}}$$

$$a)\ \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n =$$

$$= \lim_{t\to\infty}\left(1-\frac{1}{t}\right)^{t-1} = \lim_{t\to\infty}\frac{\left(1-\frac{1}{t}\right)^t \to \frac{1}{e}}{(1-\underbrace{\tfrac{1}{t}}_{\to 0})\to 1} = \underline{\underline{\frac{1}{e}}}$$

$n+1=t$
za $n\to\infty$ gre $t\to\infty$

$t=n+1 \quad (n=t-1)$
$2n-1 = 2(t-1)-1 = 2t-3$
za $n\to\infty$ gre $t\to\infty$

$$b)\ \lim_{n\to\infty}\left(2-\frac{n}{n+1}\right)^{2n-1} = \lim_{n\to\infty}\left(1+1-\frac{n}{n+1}\right)^{2n-1} = \lim_{n\to\infty}\left(1+\frac{n+1-n}{n+1}\right)^{2n-1} = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{2n-1} =$$

$$= \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{2t-3} = \lim_{t\to\infty}\frac{\left(1+\frac{1}{t}\right)^{2t}}{(\underbrace{1+\tfrac{1}{t}}_{1})^3 \to 1^3} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{2t} = \lim_{t\to\infty}\left(\left(1+\frac{1}{t}\right)^t\right)^2 =$$

$$= \Big(\underbrace{\lim_{t\to\infty}\left(1+\tfrac{1}{t}\right)^t}_{\to e}\Big)^2 = \underline{\underline{e^2}}$$

$$\boxed{f \text{ zvezna funkcija} \Rightarrow \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)}$$

$\infty-\infty = ?$

$$c)\ \lim_{n\to\infty} n\left(\log(n+1)-\log n\right) = \lim_{n\to\infty} n\log\frac{n+1}{n} = \lim_{n\to\infty}\log\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)^n =$$

$$= \log\Big(\underbrace{\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n}_{e}\Big) = \log(e) = \underline{\underline{1}}$$

$\log = \log_e = \ln$

$\frac{\infty}{\infty} = ?$

$$d)\ \lim_{n\to\infty}\frac{n^2\sin(n^2)+n-6}{n^3-n+1}\ \Big/ : n^3 = \lim_{n\to\infty}\frac{\frac{\sin(n^2)}{n}+\frac{1}{n^2}-\frac{6}{n^3}}{1-\frac{1}{n^2}+\frac{1}{n^3}} = \lim_{n\to\infty}\frac{\overbrace{\sin(n^2)}^{\text{omejena}}}{n} = \underline{\underline{0}}$$

$\frac{1}{n^2}\to 0,\ \frac{6}{n^3}\to 0,\ \frac{1}{n^2}\to 0,\ \frac{1}{n^3}\to 0,\ n\to\infty$

$$\boxed{-m \le f(n) \le M \quad \forall n \text{ in } \lim_{n\to\infty} g(n) = \infty \Rightarrow \lim_{n\to\infty}\frac{f(n)}{g(n)} = 0}$$

4. * Podani sta vrsti

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} \quad \text{ter} \quad \sum_{n=0}^{\infty} (-1)^n n.$$

Za vsako izmed omenjenih vrst:

(a) ugani in dokaži formulo za $n-$to delno vsoto;

(b) po definiciji izračunaj vsoto vrste, če obstaja.

$$\sum_{n=1}^{\infty} \underbrace{\frac{2}{n(n+2)}}_{a_n} = s \qquad a_n = \frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2)+Bn}{n(n+2)} = \frac{n(A+B)+2A}{n(n+2)}$$

$$2 = n(A+B) + 2A$$

$$n \cdot 0 + 2 = n(A+B) + 2A$$

$$\left.\begin{array}{l} A+B=0 \\ 2A = 2 \end{array}\right\} A=1,\ B=-1 \rightsquigarrow a_n = \frac{1}{n} - \frac{1}{n+2}$$

$$\boxed{\begin{array}{l} s_n = a_0 + a_1 + \cdots + a_n \quad \text{delne vsote} \\ s = \sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n \end{array}}$$

$$s_1 = a_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$s_2 = a_1 + a_2 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{2}{3} + \frac{1}{4} = \frac{8+3}{12} = \frac{11}{12}$$

$$s_3 = a_1 + a_2 + a_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{20+10-5-4}{20} = \frac{21}{20}$$

$$s_4 = a_1 + a_2 + a_3 + a_4 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} = \frac{30+15-6-5}{30} = \frac{34}{30}$$

$\vdots$ ostanejo prvi, tretji, predpredzadnji in zadnji, ostali se odštejejo

$$s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{3}{2} - \underbrace{\frac{1}{n+1}}_{\to 0} - \underbrace{\frac{1}{n+2}}_{\to 0} \right) = \frac{3}{2}$$

$$\sum_{n=0}^{\infty} (-1)^n n = s \qquad a_n = (-1)^n n$$

$$s_0 = a_0 = 0$$

$$s_1 = a_0 + a_1 = 0 + (-1) = -1$$

$$s_2 = a_0 + a_1 + a_2 = s_1 + a_2 = -1 + 2 = 1$$

$$s_3 = s_2 + a_3 = 1 + (-3) = -2$$

$$s_4 = s_3 + a_4 = -2 + 4 = 2$$

$$s_5 = s_4 + a_5 = 2 + (-5) = -3$$

$$s_6 = s_5 + a_6 = -3 + 6 = 3$$

$\vdots$

$$s_n = \begin{cases} \frac{n}{2}; & n \text{ sod} \\ -\frac{n+1}{2}; & n \text{ lih} \end{cases} \qquad \left.\begin{array}{l} \lim_{n\to\infty} \frac{n}{2} = \infty \\ \lim_{n\to\infty} \left(-\frac{n+1}{2}\right) = -\infty \end{array}\right\} \lim_{n\to\infty} s_n \text{ ne obstaja, vrsta ni konvergentna.}$$

5. Izračunaj vsoto naslednjih geometrijskih vrst.

(a) * $\sum_{n=1}^{\infty} \frac{10}{3^n}$

(b) $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}}$

(c) * $3/2 + 1 + 2/3 + 4/9 + 8/27 + \dots$

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3 \cdot 2^{3n-2}}$

(e) * $\sum_{n=1}^{\infty} (\frac{x}{2})^{3n}$ za tiste $x \in \mathbb{R}$, za katere vrsta konvergira

$$|q| < 1 \Rightarrow$$

- $1 + q + q^2 + \dots + q^k = \frac{1-q^{k+1}}{1-q}$
- $1 + q + q^2 + \dots = \frac{1}{1-q}$
- $q^k + q^{k+1} + q^{k+2} + \dots = \frac{q^k}{1-q}$

(a) * $\sum_{n=1}^{\infty} \frac{10}{3^n} = 10 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 10 \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} = 10 \cdot \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{10}{2} = \underline{\underline{5}}$

$q = \frac{1}{3}, k = 1$

(b) $\sum_{n=2}^{\infty} \frac{2^n}{3^{2n-1}} = \sum_{n=2}^{\infty} 3 \cdot \frac{2^n}{3^{2n}} = 3 \sum_{n=2}^{\infty} \left(\frac{2}{9}\right)^n = 3 \cdot \frac{\left(\frac{2}{9}\right)^2}{1-\frac{2}{9}} = 3 \cdot \frac{\frac{4}{81}}{\frac{7}{9}} =$

$k = 2$, $q = 2/9$

$= \frac{3 \cdot 4 \cdot 9}{7 \cdot 81} = \frac{4}{7 \cdot 3} = \underline{\underline{\frac{4}{21}}}$

(c) * $3/2 + 1 + 2/3 + 4/9 + 8/27 + \dots = \frac{3}{2} + \frac{3}{2} \cdot \frac{2}{3} + \frac{3}{2} \cdot \left(\frac{2}{3}\right)^2 + \dots = \frac{3}{2}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right) = \frac{3}{2} \cdot \frac{1}{1-\frac{2}{3}} =$

$k = 0$, $q = \frac{2}{3}$

$= \frac{3}{2} \cdot \frac{1}{\frac{1}{3}} = \underline{\underline{\frac{9}{2}}}$

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3 \cdot 2^{3n-2}} = \sum_{n=1}^{\infty} \frac{1}{3} \cdot 2^2 \frac{(-2)^n}{2^{3n}} = \frac{4}{3} \sum_{n=1}^{\infty} \left(-\frac{2}{8}\right)^n = \frac{4}{3} \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{4}{3} \frac{\left(-\frac{1}{4}\right)}{1-\left(-\frac{1}{4}\right)} = \frac{4}{3} \frac{-\frac{1}{4}}{\frac{5}{4}} = \frac{4}{3} \cdot \left(-\frac{1}{5}\right) = \underline{\underline{-\frac{4}{15}}}$

$q = -\frac{1}{4}, k = 1$

(e) * $\sum_{n=1}^{\infty} (\frac{x}{2})^{3n}$, za tiste $x \in \mathbb{R}$, za katere vrsta konvergira.

$\sum_{n=1}^{\infty} (\frac{x}{2})^{3n} = \sum_{n=1}^{\infty} \left(\left(\frac{x}{2}\right)^3\right)^n = \sum_{n=1}^{\infty} \left(\frac{x^3}{8}\right)^n = \frac{\frac{x^3}{8}}{1-\frac{x^3}{8}} = \frac{\frac{x^3}{8}}{\frac{8-x^3}{8}} = \underline{\underline{\frac{x^3}{8-x^3}}}$

za $q = \frac{x^3}{8}$, $k = 1$, če je $\left|\frac{x^3}{8}\right| < 1$

$\downarrow$

$\frac{|x^3|}{8} < 1$

$|x^3| < 8$

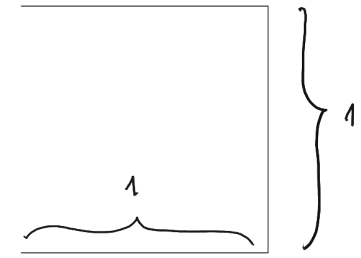
$|x|^3 < 8$

$\underline{\underline{|x| < 2}}$

Vsota je $\frac{x^3}{8-x^3}$ za $x \in (-2, 2)$. Za ostale x vrsta ne konvergira.

6. Izračunaj obseg Hilbertove krivulje:

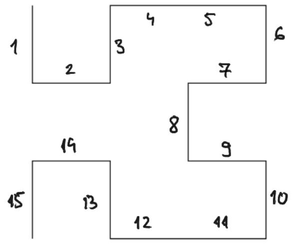
$n = 1$



3 stranice dolžine 1 $\rightsquigarrow$ $\sigma_1 = 3$ $\quad 2+1$

$4-1 \qquad \frac{1}{2-1}$

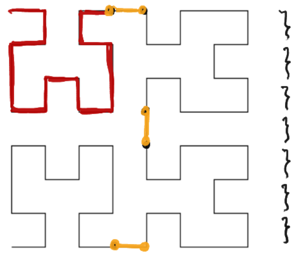
$n = 2$



15 stranic dolžine $\frac{1}{3}$ $\rightsquigarrow$ $\sigma_2 = 15 \cdot \frac{1}{3} = 5$ $\quad 4+1$

$16-1 \qquad \frac{1}{4-1}$

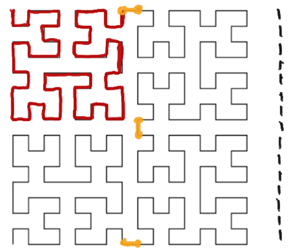
$n = 3$



$3 + 4 \cdot 15 = 63$ stranic dolžine $\frac{1}{7}$ $\rightsquigarrow$ $\sigma_3 = 63 \cdot \frac{1}{7} = 9$ $\quad 8+1$

$64-1 \qquad \frac{1}{8-1}$

$n = 4$



$3 + 4 \cdot 63 = 255$ stranic dolžine $\frac{1}{15}$ $\rightsquigarrow$ $\sigma_4 = 255 \cdot \frac{1}{15} = 17$ $\quad 16+1$

$256-1 \qquad \frac{1}{16-1}$

n=1 n=2 n=3 n=4

$3, 15, 63, 255, \ldots$ $\quad (4^n - 1)$ stranic ✓

n=1 n=2 n=3 n=4

$1, \frac{1}{3}, \frac{1}{7}, \frac{1}{15} \ldots$ $\quad \left(\frac{1}{2^n-1}\right)$ dolžina ✓

$$\sigma_n = (4^n - 1) \cdot \frac{1}{2^n - 1}$$

$$\sigma_n = \frac{4^n - 1}{2^n - 1} = \frac{(2^n)^2 - 1}{2^n - 1} = \frac{(2^n - 1)(2^n + 1)}{2^n - 1} = 2^n + 1 \checkmark$$

$$\sigma = \lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} (2^n + 1) = \infty$$

★ Na vsakem koraku se prejšnji vzorec ponovi v 4 pomanjšanih kopijah, povezanih s 3 dodatnimi robovi.

s_n ... število stranic na n-tem koraku

d_n ... dolžina stranice na n-tem koraku

$s_{n+1} = 4 \cdot s_n + 3$, $s_1 = 3$, z indukcijo dokaži: $s_n = 4^n - 1$

$d_{n+1} = \frac{1}{\frac{2}{d_n} + 1}$, $d_1 = 1$, z indukcijo dokaži: $d_n = \frac{1}{2^n - 1}$

$+1 \rightarrow$ št. robov v navpični smeri

Baza: $d_1 = \frac{1}{2^1 - 1}$

$1 = \frac{1}{2-1}$ ✓

Ind. predp.: $d_n = \frac{1}{2^n - 1}$

Baza: $s_1 = 4^1 - 1$

$3 = 3$ ✓

Ind. predp. $s_n = 4^n - 1$

$s_{n+1} = 4 \cdot s_n + 3 = 4(4^n - 1) + 3 = 4^{n+1} - 4 + 3 = 4^{n+1} - 1$ ✓

Ind. korak: $d_{n+1} = \frac{1}{\frac{2}{d_n} + 1} = \frac{1}{\frac{2}{\frac{1}{2^n-1}} + 1} = \frac{1}{2(2^n-1)+1} = \frac{1}{2^{n+1} - 1}$ ✓