Osnove matematične analize
Vaje 1

1. * Z matematično indukcijo dokaži:

   (a) $1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$,

   (b) $1^3 + 2^3 + \ldots + n^3 = n^2(n+1)^2/4$,

   (c) $n! > 2^{n-1}$ za $n > 2$,

   (d) $1 \cdot 4 + 2 \cdot 4^2 + 3 \cdot 4^3 + \cdots + n \cdot 4^n > \frac{(3n-1)4^{n+1}}{9}$.

a) Baza: $n=1$ $\quad 1 \cdot 2 = \frac{1}{3} \cdot 1 \cdot 2 \cdot 3$?

$2 = 2$? ✓

Indukcijski korak: $n \to n+1$

Vemo: $1 \cdot 2 + \cdots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$ indukcijska predpostavka

Radi bi videli: $1 \cdot 2 + \cdots + n(n+1) + (n+1)(n+2) \overset{?}{=} \frac{1}{3}(n+1)(n+2)(n+3)$

$1 \cdot 2 + \cdots + n(n+1) + (n+1)(n+2) \overset{\text{ind. predp.}}{=} \frac{1}{3} n(n+1)(n+2) + (n+1)(n+2) =$

$= (n+1)(n+2)\left(\frac{1}{3}n + \underbrace{1}_{3 \cdot \frac{1}{3}}\right) = \frac{1}{3}(n+1)(n+2)(n+3)$ ✓

1/3

b) Baza: $n=1$ $\quad 1^3 = 1^2 \cdot 2^2/4$?

$1 = 1$? ✓

Indukcijski korak: $n \to n+1$

Vemo: $1^3 + \cdots + n^3 = n^2(n+1)^2/4$ indukcijska predpostavka

Radi bi videli: $1^3 + \cdots + n^3 + (n+1)^3 \overset{?}{=} (n+1)^2(n+2)^2/4$

$1^3 + \cdots + n^3 + (n+1)^3 \overset{\text{ind. predp.}}{=} \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + \underbrace{(n+1)}_{\frac{1}{4} \cdot 4(n+1)}\right) = (n+1)^2 \cdot \frac{1}{4}\underbrace{(n^2 + 4n + 4)}_{(n+2)^2}$

$\frac{1}{4}$

$= \frac{1}{4}(n+1)^2(n+2)^2$ ✓

c) Baza: $n=3$ $\quad 3! > 2^{3-1}$?

$6 > 4$? ✓

Indukcijski korak: $n \to n+1$

Vemo: $n! > 2^{n-1}$ indukcijska predpostavka

Radi bi videli: $(n+1)! > 2^n$

vsi neenačaji v pravo smer!

$(n+1)! = n! \cdot (n+1) \overset{\text{ind. predp.}}{>} 2^{n-1} \cdot \underbrace{(n+1)}_{>2} > 2^{n-1} \cdot 2 = 2^n$ ✓

d) Baza: $n=1$ $\quad 1\cdot 4 > \frac{(3\cdot 1-1)\cdot 4^2}{9}$?

$4 > \frac{32}{9}$?

$4 > 3+\frac{5}{9}$? ✓

Indukcijski korak: $n \longrightarrow n+1$

Vemo: $1\cdot 4 + \dots + n\cdot 4^n > \frac{(3n-1)\cdot 4^{n+1}}{9}$ indukcijska predpostavka

Radi bi videli: $1\cdot 4 + \dots + n \cdot 4^n + (n+1)\cdot 4^{n+1} > \frac{(3n+2)\cdot 4^{n+2}}{9}$

$$1\cdot 4 + \dots + n\cdot 4^n + (n+1)\cdot 4^{n+1} \overset{\text{ind. predp.}}{>} \frac{(3n-1)\cdot 4^{n+1}}{9} + \underbrace{(n+1)\cdot 4^{n+1}}_{\cdot 9 \cdot \frac{1}{9}} = \frac{4^{n+1}}{9}\Big(3n-1+9(n+1)\Big) =$$

$$= \frac{4^{n+1}}{9}\left(3n-1+9n+9\right) = \frac{4^{n+1}}{9}\left(12n+8\right) = \frac{4^{n+2}}{9}\left(3n+2\right)$$ ✓

2. * Z uporabo matematične indukcije utemelji, da za vsako naravno število $n \geq 2$ velja:

$$\log\left(1-\frac{1}{2^2}\right)+\log\left(1-\frac{1}{3^2}\right)+\cdots+\log\left(1-\frac{1}{n^2}\right)=\log\left(\frac{n+1}{2n}\right).$$

$$\boxed{\log a + \log b = \log(ab)}$$

$$\log\left(1-\frac{1}{2^2}\right)+\cdots+\log\left(1-\frac{1}{n^2}\right) = \log\left(\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)\right) \overset{?}{=} \log\left(\frac{n+1}{2n}\right)$$

$$\boxed{\log a = \log b \iff a = b}$$

$\Rightarrow$ Ali je $\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$?

Baza: $n=2$ $\quad 1-\frac{1}{2^2} = \frac{3}{4}$?

$\frac{3}{4} = \frac{3}{4}$? ✓

Indukcijski korak: $n \to n+1$

Vemo: $\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ indukcijska predpostavka

Radi bi videli: $\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2(n+1)}$

$$\underbrace{\left(1-\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)}_{\text{ind. predp.}}\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\cdot\left(1-\frac{1}{(n+1)^2}\right) = \frac{\cancel{n+1}}{2n}\cdot\frac{(n+1)^2-1}{(n+1)^{\cancel{2}}} = \frac{n^2+2n}{2n(n+1)} = \frac{n+2}{2(n+1)}$$ ✓

3. Dokaži, da je za vsako naravno število $n > 0$ število $11^{n+1} + 12^{2n-1}$ deljivo s 133.

$a, x \in \mathbb{Z}$

a deli x (oznaka $a \mid x$) $\iff$ $x = k \cdot a$ za nek $k \in \mathbb{Z}$

Baza: $n = 1$ $\quad 11^2 + 12^{2\cdot 1 - 1} = 121 + 12 = 133 = 1 \cdot 133$ ✓

Indukcijski korak: $n \longrightarrow n+1$

Vemo: $133 \mid (11^{n+1} + 12^{2n-1})$ oz. $11^{n+1} + 12^{2n-1} = 133 \cdot k$ za nek $k \in \mathbb{Z}$

Radi bi videli: $133 \mid (11^{n+2} + 12^{2(n+1)-1})$

$$11^{n+2} + 12^{2(n+1)-1} = 11^{n+2} + 12^{2n+1} = 11 \cdot 11^{n+1} + 12^{2n+1} = 11\underbrace{(11^{n+1} + \overset{+}{12^{2n-1}})}_{\text{ind. predp.}} \overset{-}{-} 11 \cdot 12^{2n-1} + 12^{2n+1} =$$

$$11 \cdot 133 \cdot k + 12^{2n-1}(\underbrace{-11 + 12^2}_{144-11}) = 133 \cdot 11k + 133 \cdot 12^{2n-1} = 133(11k + 12^{2n-1})$$ ✓

4. * Dokaži, da je za vsako naravno število n število $7^{n+2} + 8^{2n+1}$ deljivo s 57.

Baza: $n=0$ $\quad 7^2 + 8^1 = 49 + 8 = 57$, $\quad 57 \mid 57$ ✓

Indukcijski korak: $n \to n+1$

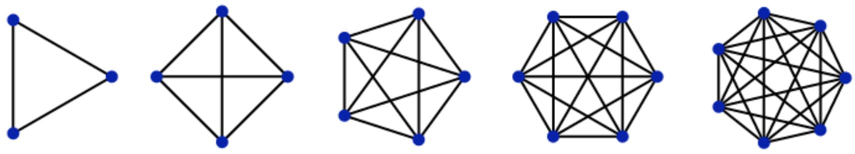
Vemo: $7^{n+2} + 8^{2n+1} = 57 \cdot k$ za nek $k \in \mathbb{Z}$

Radi bi videli: $57 \mid 7^{n+3} + 8^{2n+3}$

$$7^{n+3} + 8^{2n+3} = 7 \cdot 7^{n+2} \overset{+}{+} 7 \cdot 8^{2n+1} \overset{-}{-} 7 \cdot 8^{2n+1} + 8^{2n+3} = 7\left(\underbrace{7^{n+2} + 8^{2n+1}}\right) + 8^{2n+1}\left(\underbrace{-7 + 8^2}_{64-7}\right) =$$

$$= 7 \cdot 57k + 8^{2n+1} \cdot 57 = 57\left(7k + 8^{2n+1}\right)$$ ✓

5. Ugani formulo za število diagonal konveksnega mnogokotnika in jo dokaži z matematično indukcijo.



d_n = # diagonal v konveksnem n-kotniku

$d_3 = 0 \quad d_4 = 2 \quad d_5 = 5 \quad d_6 = 9 \quad d_7 = \dots$

Brez indukcije: Vseh možnih povezav je $\binom{n}{2} = \frac{n(n-1)}{2}$. Od tega je n stranic, torej je diagonal $\frac{n(n-1)}{2} - n = \frac{n^2-n-2n}{2} = \frac{n^2-3n}{2} = \frac{n(n-3)}{2}$.

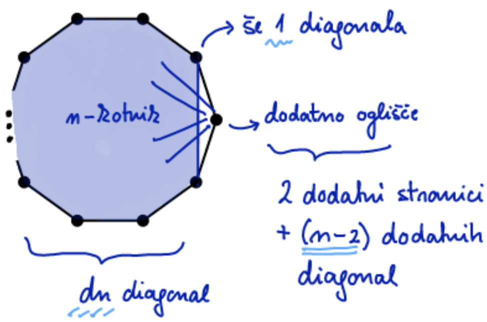
Z indukcijo: konveksni n-kotnik ima $\frac{n(n-3)}{2}$ diagonal

Baza: $n=3 \qquad d_3 = 0 = \frac{3\cdot(3-3)}{2}$ ✓

Indukcijski korak: $n \to n+1$

Vemo: $d_n = \frac{n(n-3)}{2}$

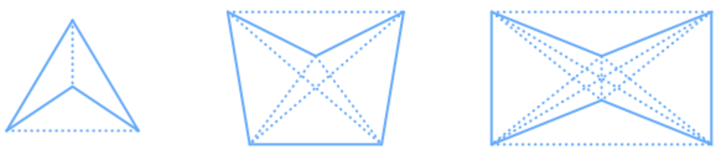
Radi bi videli: $d_{n+1} = \frac{(n+1)(n-2)}{2}$



$$d_{n+1} = d_n + n-2+1 = \frac{n(n-3)}{2} + n - 1 = \frac{n^2-3n+2n-2}{2}$$

$$= \frac{n^2-n-2}{2} = \frac{(n+1)(n-2)}{2} \checkmark$$

Za nekonveksne še vedno velja, le da diagonale ne ležijo nujno v notranjosti.



6. * Za vsako od naslednjih množic določi infimum in supremum. Ali obstaja minimum ali maksimum?

(a) $A = \{x \in \mathbb{R}\,;\ ||x-1|-2| \geq 1\}$,

(b) $B = \{x \in \mathbb{R}\,;\ ||x-1|-2| \geq 1, x \leq 5 \text{ in } x > 1\}$,

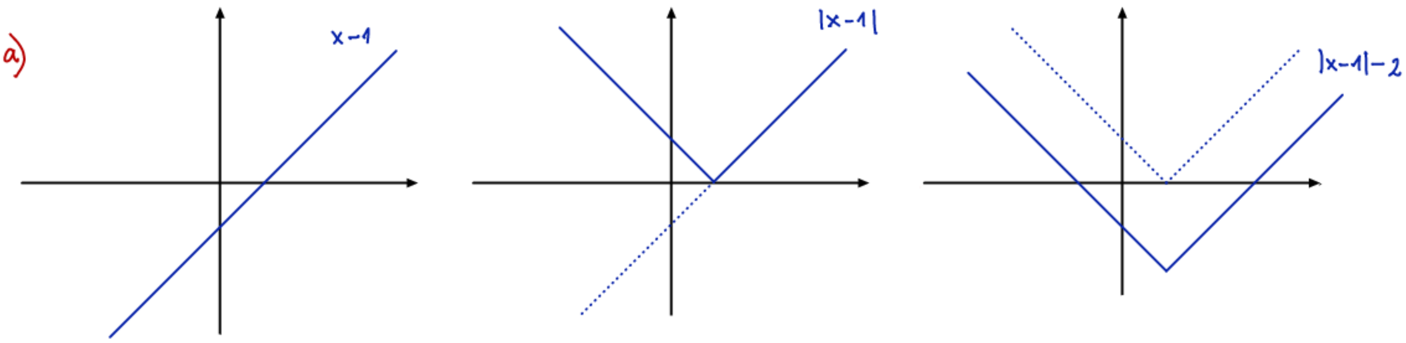
(c) $C = \{2 + \sin x\,;\ x \in \mathbb{R}\}$,

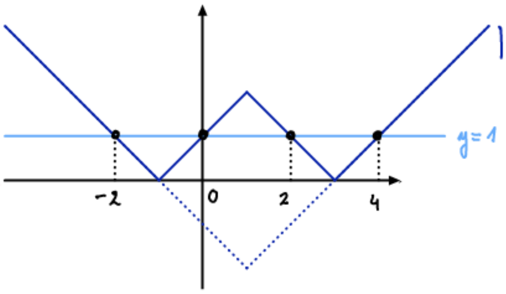
(d) $D = \{x \in \mathbb{R}\,;\ \log 2 + \log(x^2-1) \leq 2\log|x-1|\}$,

(e) $E = \{x \in \mathbb{R}\,;\ \log 2 + \log|x^2-1| \leq 2\log|x-1|\}$,

(f) $F = \{x \in \mathbb{R}\,;\ \log 2 + \log(x^2-1) \leq 2\log(x-1)\}$.

a)



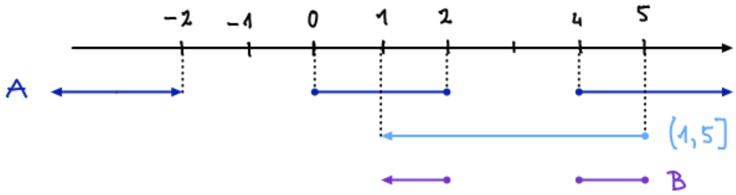


$||x-1|-2| \geq 1$ na $(-\infty,-2] \cup [0,2] \cup [4,\infty) = A$

$\sup A = \infty$, max ne obstaja

$\inf A = -\infty$, min ne obstaja

b) $B = A \cap (-\infty,5] \cap (1,\infty) = \big((-\infty,-2] \cup [0,2] \cup [4,\infty)\big) \cap \underbrace{(-\infty,5] \cap (1,\infty)}_{(1,5]} =$



$= \big((-\infty,-2] \cup [0,2] \cup [4,\infty)\big) \cap (1,5] =$

$= \underbrace{(-\infty,-2] \cap (1,5]}_{\emptyset} \cup \underbrace{[0,2] \cap (1,5]} \cup \underbrace{[4,\infty) \cap (1,5]} =$

$= \underline{\underline{(1,2] \cup [4,5]}} = B$

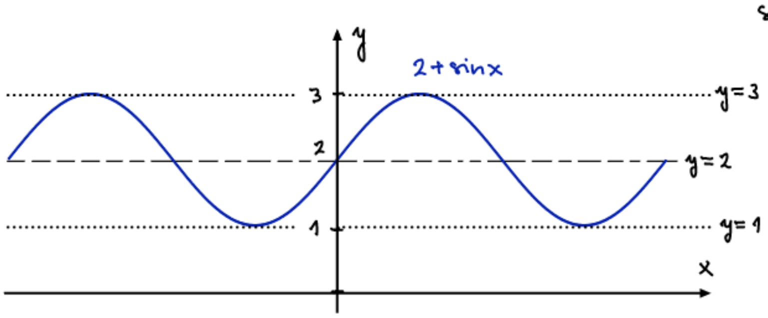
$\sup B = \max B = 5$

$\inf B = 1$, min ne obstaja

c) $-1 \leq \sin x \leq 1 \implies 1 \leq 2 + \sin x \leq 3$

$\inf C = \min C = 1$

$\sup C = \max C = 3$



d) $\log 2 + \log(x^2-1) = \log 2(x^2-1)$
$2\log|x-1| = \log|x-1|^2$
$\Rightarrow \log 2(x^2-1) \leq \log|x-1|^2$

$\log a \leq \log b \Rightarrow a \leq b$, ker je log naraščajoča funkcija!

$2(x^2-1) \leq |x-1|^2$

$2(x-1)(x+1) \leq |x-1|^2$

$|x-1| = \begin{cases} x-1 ; & x \geq 1 \\ 1-x ; & x < 1 \end{cases}$

def. območje $\log : (0,\infty) = \mathbb{R}^+$

- $x^2-1>0 \Rightarrow x^2>1 \Rightarrow |x|>1 \Rightarrow (-\infty,-1)\cup(1,\infty)$ ★

i) $x>1$

$2(x-1)(x+1) \leq (x-1)^2 \quad /:(x-1)>0$

$2(x+1) \leq x-1$

$2x+2 \leq x-1$

$x \leq -3 \quad \emptyset$

ii) $x=1$

$0 \leq 0$ ✓

$\{1\}$

iii) $x<1$

$2(x-1)(x+1) \leq (1-x)^2$

$2(x-1)(x+1) \leq (x-1)^2 \quad /:(x-1)<0$

$2(x+1) \geq x-1$

$2x+2 \geq x-1$

$x \geq -3 \quad [-3,1)$

$D = [-3,1] \cap \big((-\infty,-1)\cup(1,\infty)\big)$ ★ $= [-3,-1)$

$\inf D = \min D = -3$
$\sup D = -1$, max ne obstaja

e) $\log 2 + \log|x^2-1| = \log 2|x-1||x+1|$
$2\log|x-1| = \log|x-1|^2$

$\log 2|x-1||x+1| \leq \log|x-1|^2$

$2|x-1||x+1| \leq |x-1|^2 \quad /:|x-1|\geq 0$

$2|x+1| \leq |x-1|$

$|x^2-1|>0$ za $x \neq \pm 1$
$|x-1|>0$ za $x \neq 1$
def. območje je $\mathbb{R}\setminus\{-1,1\}$

$|x+1| = \begin{cases} x+1 ; & x \geq -1 \\ -x-1 ; & x<-1 \end{cases}$

$|x-1| = \begin{cases} x-1 ; & x \geq 1 \\ 1-x ; & x<1 \end{cases}$

i) $x \geq 1$

$2(x+1) \leq x-1$

$x \leq -3$

$\emptyset$

ii) $-1 \leq x < 1$

$2(x+1) \leq 1-x$

$3x \leq -1$

$x \leq -\frac{1}{3}$

$[-1,-\frac{1}{3}]$

iii) $x<-1$

$2(-x-1) \leq 1-x$

$-x \leq 3$

$x \geq -3$

$[-3,-1)$

$E = \big([-3,-1)\cup[-1,-\frac{1}{3}]\big) \cap \big(\mathbb{R}\setminus\{-1,1\}\big) = [-3,-\frac{1}{3}]\setminus\{-1,1\} = [-3,-1)\cup(-1,-\frac{1}{3}]$

$\inf E = \min E = -3$
$\sup E = \max E = -\frac{1}{3}$

f) $\ldots \Rightarrow 2(x^2-1) \leq (x-1)^2 \longrightarrow 2(x-1)(x+1) \leq (x-1)^2 \quad /:(x-1)>0$

$2(x+1) \leq x-1$

$x \leq -3 \quad (-\infty,-3]$

def. območje log:
$x^2-1>0 \Rightarrow (-\infty,-1)\cup(1,\infty)$
$x-1>0 \Rightarrow (1,\infty)$
$\Big\}\ x \in (1,\infty)$

$F = (-\infty,-3] \cap (1,\infty) = \emptyset$

$\inf \emptyset = \infty$
$\sup \emptyset = -\infty$

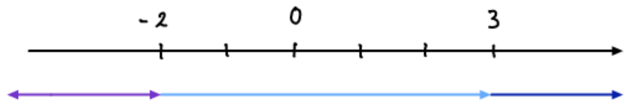
$\min \emptyset$ in $\max \emptyset$ ne obstajata

7. Poišči množico rešitev neenačbe

$$|x+2| - |x-3| > 3.$$

Določi še njen infimum in supremum. Ali ima minimum? Maksimum?

$$|x+2| = \begin{cases} x+2\ ; & x \geq -2 \\ -x-2\ ; & x < -2 \end{cases} \qquad |x-3| = \begin{cases} x-3\ ; & x \geq 3 \\ 3-x\ ; & x < 3 \end{cases}$$



i) $x \in (-\infty, -2)$

$$-x-2-(3-x) > 3$$

$$-5 > 3 \ //$$

$$\emptyset$$

ii) $x \in [-2, 3)$

$$x+2-(3-x) > 3$$

$$2x-1 > 3$$

$$x > 2 \rightarrow (2, \infty)$$

$$(2,\infty) \cap [-2,3) = (2,3)$$

iii) $x \in [3, \infty)$

$$x+3-(x-3) > 3$$

$$6 > 3 \ \checkmark$$

$$[3, \infty)$$

$$\Rightarrow \quad x \in (2,3) \cup [3,\infty) = (2,\infty)$$

inf : 2   sup : ∞ min, max ne obstajata