Area of a triangle with vertices on three circles

You are given three circles in the plane $\mathbb{R}^2$: K_1, K_2, and K_3. A triangle ABC has vertex A on the circle K_1, vertex B on K_2 and vertex C on K_3. The objective is to find the triangle with the largest area among all such triangles ABC. Denote by (p_i, q_i) the coordinates of the centre of the circle K_i, and use $r_i > 0$ to denote the radius of the circle K_i. Write an Octave function which, for three circles given by $[p_i, q_i, r_i]^\mathsf{T}$, finds a triangle ABC with largest possible area.

Task

1. To solve the task find the *minimum* of an appropriate function of three variables using the gradient descent method. Advice: Use the subtasks below to efficiently determine the gradient of the appropriate function. (You can, of course, suitably adjust the use of the gradient descent method...)

   (a) Let $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ be parametrizations of three (for now arbitrary) plane curves. Write down a formula for the area of a triangle with vertices on these three curves. Position vectors of vertices of this triangle are therefore $\mathbf{p}_1(t)$, $\mathbf{p}_2(u)$, and $\mathbf{p}_3(v)$, the area of this triangle is a function of three variables, denote it by $f(t, u, v)$.

   (b) Express the gradient of $-f^2$ using parametrizations $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ and their derivatives $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$, and $\dot{\mathbf{p}}_3$. Hint: Chain rule.

   (c) Write down a parametrization of a circle with radius r and centre at the point (p, q).

   (d) What is the expression for $\mathrm{grad}(-f^2)$ in case $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are parametrizations of three circles? Hint: Chain rule again.

2. Find a local minimum of $-f^2$ using the gradient descent method. Use the parameters t, u, and v, at which a minimum of $-f^2$ is attained, to evaluate the position vectors of a triangle with the largest area and use those to evaluate the actual area.

3. Write an Octave function `[T, pl] = trikotnik(K)`, which returns the position vectors of the vertices of the triangle `T` and the area of the triangle `pl`.

   (a) `T` is a 2×3 matrix of position vectors of vertices of the triangle, the first vertex on the first circle, the second on the second and the third on the third. The coordinates of the vertices should be evaluated to **8 decimal places** (absolute error).

   (b) `pl` is the area of the triangle.

   (c) `K = [k1, k2, k3]` is a 3×3 matrix with columns od the form $\mathbf{k}_i = [p_i, q_i, r_i]^\mathsf{T}$, where r_i is the radius and (p_i, q_i) the centre of the circle K_i.

4. Equip the file `trikotnik.m` with comments, a test and a demo. For the test: Find a suitable configuration of circles for which you can evaluate (or guess) the

solution directly (by hand). The demo should draw all three circle and a triangle with vertices on these circles with largest possible area.

Use your student ID to obtain the data for the demo: first three digits should be p_1, p_2, p_3, the last three digits should be q_1, q_2, q_3, i^{th} radius is then $r_i = p_i + q_i + 1$. (Your student ID is therefore $\mathsf{p_1p_2p_3**q_1q_2q_3}$.)

Warning: The method just described is sensitive to the orientation of the triangle, ie. the choice of the order of the vertices of the triangle. In general this cannot be avoided since the final solution may be a triangle with either positive or negative orientation. Nevertheless, the function `trikotnik` must return a triangle with the largest area. Think about how you will solve this problem. (*Hint:* Changing the order of any two vertices will change the orientation of the triangle.)

Submission

Use the online classroom to submit the following:

1. the file **trikotnik.m** which should be well-commented, contain at least one test and a demo,
2. a report file **solution.pdf** which contains the necessary derivations and answers to questions.

While you can discuss solutions of the problems with your colleagues, the programs and report must be your own creation. You can use all Octave functions from problem sessions.